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THESIS

A NUMERICAL ANALYSIS OF PIPE FLOW STABILITY

by

David Bruce Wallace

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Numerical results for angular wave numbers zero and six show that the flow is stable at all Reynolds numbers. Results for angular wave number one contain instabilities at all Reynolds numbers for small values of the axial wave number. These results are tabulated, plotted, and discussed in detail in this paper.

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A Numerical Analysis of Pipe Flow Stability

by

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TABLE OF SYMBOLS

D	Differential operator $\partial/\partial r$
D^2, D^3, D^4	Higher order partial derivatives with respect to r.
e	Base of natural logrithms.
$\vec{e}_x$, $\vec{e}_r$, $\vec{e}_\theta$	Unit vectors along the x, r, and θ axes in cylindrical coordinates.
F, G, H	Components of the velocity vector potential function defined in equation (2-28).
h	The increment of the radius in the finite difference equations, $h = 1/N$.
i	$+\sqrt{-1}$, the imaginary unit.
i	Used as an index to distinguish the transformed coefficient matrices $[M_i']$ of equation (2-43) and in the discrete finite difference equations.
j	Used as an index to distinguish the transformed coefficient matrices $[N_j']$ of equation (2-43).
N	The number of interior points along the pipe radius used in the finite difference mesh.
n	Angular wave number of the perturbation in the θ direction, where $n = 0, 1, 2, 3 \ldots\ldots$
O	Symbol denoting the phrase "of order".
P	Pressure appearing in equations (2-4, 5, 11, and 12).

P	Component of the velocity vector potential derived from the component G after a change of variable.
Q	Component of the velocity vector potential derived from the component H after a change of variable.
R	Normalized pipe radius, $R = 1$.
$\bar{R}$	The dimensional pipe radius used to define Re in equation (2-6).
Re	Reynolds number based on pipe radius and mean volumetric velocity, as defined in equation (2-6).
t	Time.
U	The axial component of the mean dimensionless velocity of the flow as defined in equation (2-13).
$\bar{U}$	The mean dimensional volumetric velocity of the flow used to define Re in equation (2-6).
u, v, w	Components of the complex perturbation velocity.
$\vec{V}$	Velocity vector of the mean flow.
$\vec{V}'$	Velocity vector of the total or resultant flow.
$\vec{v}$	Complex perturbation velocity vector.
$\vec{W}$	The velocity vector potential function defined in equation (2-28).
x, r, θ	Cylindrical coordinates.
X	Symbol for $i\alpha x + in\theta + \gamma t$, defined in equation (2-29).

$\{X^*\}$	Eigenvector corresponding to the least stable eigenvalue.
α	The real axial wave number of the perturbation in the x direction, where $\alpha \geqq 0$.
γ	The complex frequency of the perturbation defined in equation (2-30).
γ^*	The maximum algebraic value of the real part of the complex frequency which is also the least stable eigenvalue, also referred to as GAMMA*.
$\vec{\Gamma}$	Shorthand notation of the vorticity transport equations as defined in equation (2-32).
$\Gamma_x, \Gamma_r, \Gamma_\theta$	Components of $\vec{\Gamma}$ in cylindrical coordinates.
ε	The residual error of the least stable eigenvalue solution of the governing equations.
υ	Kinematic viscosity.
μ	Viscosity.
ρ	Density.
$\vec{\Omega}$	Mean vorticity vector.
$\vec{\Omega}'$	Total or resultant vorticity vector.
$\vec{\omega}$	Perturbation vorticity vector.
∇	Linear vector operator (nabla).
$\times$	Vector cross-product operator.
$\nabla\times$	Vector curl operator.

$\nabla\cdot$	Vector dot product operator.
[]	Brackets enclosing a matrix.
{}	Brackets enclosing a column vector.

I. INTRODUCTION

Osborne Reynolds [Ref. 1] conducted his classical experiments on the transition from laminar to turbulent flow of fluids in circular pipes nearly 100 years ago. Based on his experiments the critical Reynolds number for pipe flow was found to be 2300. An analytical solution for the problem of predicting instabilities of fully developed, three dimensional, incompressible flow of constant viscosity fluids in pipes has been pursued actively since then. Several approaches have been taken in solving the inherently nonlinear Navier-Stokes equations, which along with the continuity equation, govern the behavior of fluid flow in pipes of circular cross-section.

Salwen and Grosch [Ref. 2] studied pipe flow with purely sinusoidal streamwise (axial) perturbations. Infinitesimal velocity and pressure perturbations, which were explicit functions of time and not complex or exponential in form, were introduced into the flow field. Numerical calculations were carried out at various angular wave numbers and it was concluded that the flow was stable for all axial wave numbers and Reynolds numbers. Perturbations with exponential growth in space but a purely sinusoidal variation in time were explored by Garg and Rouleau [Ref. 3] and found to be stable. Gill [Ref. 4] studied combinations of exponential growth in space and in time using power series analysis and concluded that these flows were all stable.

Because of the general agreement that pipe flow is stable to infinitesimal disturbances, two other approaches to the problem of predicting instabilities at the experimentally observed critical Reynolds number have been pursued. First, investigations by Davy and Drazin [Ref. 5] confirmed that pipe Poiseuille flow was stable at all Reynolds numbers for infinitesimal disturbances that were both temporally and axially complex and exponential in nature. However, despite the fact that the governing equations must necessarily remain nonlinear with the introduction of disturbances of finite amplitude, they concluded that the flow is unstable with respect to finite disturbances. Similar theoretical examinations of plane Poiseuille flow between two parallel plates led McIntire and Lin [Ref. 6] to this same conclusion. Second, it was postulated by Huang and Chen [Ref. 7] and Leite [Ref. 8] that the origin of flow instabilities occur in the entrance region of the pipe, where the flow has not yet become fully developed. Both theoretical and experimental studies have been done to support these hypotheses. Garg [Ref. 9] also found that flow instabilities existed near the entry region for developing pipe flow. He concluded that results using the Hornbeck velocity profile more nearly approximated experimentally determined values of the critical Reynolds number. While these investigations have shown instabilities to exist in pipe flow, a completely generalized solution to the linearized problem of fully developed pipe flow has never been accomplished.

Recently, a more general theory consisting of perturbations which have a fully complex exponential form with respect to time and the axial

coordinate, a purely imaginary exponential form in the angular coordinate and are three dimensional in nature have been explored. Development of this theory by Harrison [Ref. 10] and further numerical investigations by Arnold [Ref. 11] failed to produce conclusive results that instabilities exist in fully developed pipe flow. Errors in the setup of the problem and inadequate formulation of the boundary conditions at the pipe axis contributed to these erroneous results. Arnold, however, was on the right track in his research by incorporating newly formulated boundary conditions at the axis developed by Gawain [Ref. 12].

Arnold's work, for the most part, remains valid except that he overlooked a small but significant detail that resulted in unclear definitions of stability and instability. He used a complex axial wave number α, of the form $\alpha = \alpha_R + i\alpha_I$, whereas the correct version should simply be a purely imaginary quantity of the form $i\alpha$. Thus, Arnold introduced an extra degree of freedom, namely the fictitious and incorrect quantity α_R. This accounts for the erroneous instabilities which he obtained in his numerical investigation of pipe flow.

Further work by Gawain [Ref. 13] refined the boundary conditions at the axis and incorporated the previously defined imaginary axial wave number, in exponential form, which represents only sinusoidal oscillations with respect to the axial coordinate. Using these advancements in the linearized theory, Gawain showed an asymptotic trend toward neutral stability of the flow with increasing Reynolds numbers for angular wave number, $n = 0$. This paper duplicates and confirms the results for $n = 0$, that the flow is stable at all Reynolds numbers and axial wave numbers. It is further shown that fully developed pipe flow for $n = 6$

is also stable and shows the same trends as for n = 0. Instabilities are shown to exist in the numerical analysis of the vorticity transport equations for n = 1. The paradox that now appears is that for angular wave numbers n = 0 and 6 the flow is apparently stable at all Reynolds numbers while for n = 1 the flow is apparently unstable at all Reynolds numbers. Neither of these theoretical results is consistent with the known experimental fact that there exists a critical Reynolds number below which pipe flow is stable and above which it is unstable. However, the new result for n = 1, while it can hardly be called correct, is at least encouraging in that it shows for the first time that instabilities can in fact exist in the solution of the linearized vorticity transport equations for fully developed pipe flow.

In addition to implementing the improved axis boundary conditions and the advancement in the linearized theory, improved finite differencing techniques were used to approximate the vorticity transport equations along a uniform radial mesh. Highly accurate numerical solutions for the previously mentioned angular wave numbers, n = 0, 1, and 6, were obtained using double precision, complex numbers. Additionally, finite difference equations with consistent fourth order truncation error were used throughout. Details of the development of the vorticity transport equations and the central and non-central finite difference equations are discussed in later chapters. Systematic and extensive calculations remain to be accomplished for angular wave numbers, n = 2, 3, 4, 5, and n > 6. The results of these numerical analyses should prove to be fruitful in establishing a theory which

adequately explains the experimentally observed fact of the onset of flow instabilities at the critical Reynolds number.

II. THE VORTICITY TRANSPORT EQUATION

A. BASIC THEORY

The governing equations for laminar flow of fluids in a circular pipe are derived from Newton's second law of motion,

$$\vec{F} = m\,\vec{a} \tag{2-1}$$

and the conservation of mass principle through an infinitesimal control volume. Equation (2-1) results in the familiar Navier-Stokes equation, which is a vector equation in three cylindrical coordinates and in time. The equation for conservation of mass results in the continuity equation for incompressible flow and is given by,

$$\nabla \cdot \vec{V} = 0 \tag{2-2}$$

For the case at hand, the following assumptions are made,

1. The fluid is incompressible, ρ = constant
2. The fluid has constant viscosity, μ = constant
3. The flow is fully developed, U = f(r) only
4. The flow is three dimensional in nature
5. The mean flow is steady, $\partial\vec{V}/\partial t = 0$
6. The perturbation flow is unsteady, $\partial\vec{v}/\partial t \neq 0$
7. The effects of body forces are negligible

The resulting Navier-Stokes equation can be stated as the sum of the forces per unit mass is equal to the acceleration or,

$$\Sigma \frac{\text{Force}}{\text{mass}} = \text{acceleration} \tag{2-3}$$

The forces acting on the fluid are pressure forces and viscous forces. Body forces and hydrostatic forces balance out and are not a factor here.

Equation (2-3) in its dimensional form can be expressed as,

$$- \nabla^* \frac{p^*}{\rho^*} + \left(\frac{\mu^*}{\rho^*}\right) \nabla^{2*} \vec{V}^* = \frac{d\vec{V}^*}{dt^*} \tag{2-4}$$

where the starred quantities are fully dimensional.

It should be noted here that the equations presented through equation (2-18) represent the mean flow. Equation (2-4) can also be expressed in a nondimensional form as,

$$- \nabla P + \frac{1}{Re} \nabla^2 \vec{V} = \frac{d\vec{V}}{dt} \tag{2-5}$$

where Re is the Reynolds number and is defined as,

$$Re = \frac{\bar{R}\,\bar{U}}{\nu} \tag{2-6}$$

This definition of Reynolds number is based on the pipe radius $\bar{R}$ as the characteristic length, the mean volumetric velocity $\bar{U}$ as the characteristic velocity, and the kinematic viscosity ν which is constant. The three quantities on the right side of equation (2-6) are dimensional, but Re is a dimensionless quantity. On this basis, the critical Reynolds number for transition from laminar to turbulent flow becomes 1150 (vice the value of 2300 which is obtained if the Reynolds number is

defined in terms of pipe diameter). All subsequent equations presented in this paper are in nondimensional form.

Returning to the derivation of the mean flow, the acceleration term or substantial derivative, $d\vec{V}/dt$ can be expanded to the form,

$$\frac{d\vec{V}}{dt} = (\vec{V} \cdot \nabla)\,\vec{V} + \frac{\partial \vec{V}}{\partial t} \tag{2-7}$$

but $(\vec{V} \cdot \nabla)\vec{V}$ can be further expanded by a well known vector identity to,

$$(\vec{V} \cdot \nabla)\vec{V} = \nabla\,(\frac{U^2}{2}) - \vec{V} \times (\nabla \times \vec{V}) \tag{2-8}$$

Additionally, $\nabla^2\vec{V}$ from equation (2-5) can be expanded to,

$$\nabla^2\,\vec{V} = \nabla\,(\nabla \cdot \vec{V}) - \nabla \times (\nabla \times \vec{V}) \tag{2-9}$$

but $\nabla \cdot \vec{V} = 0$, since continuity must be satisfied and equaton (2-9) becomes,

$$\nabla^2\,\vec{V} = -\,\nabla \times (\nabla \times \vec{V}) \tag{2-10}$$

With appropriate substitutions, the final form of the Navier-Stokes equation becomes,

$$-\nabla P - \frac{1}{Re}\nabla \times (\nabla \times \vec{V}) = \nabla\,(\frac{U^2}{2}) - \vec{V} \times (\nabla \times \vec{V}) + \frac{\partial \vec{V}}{\partial t} \tag{2-11}$$

A more convenient form of equation (2-11) is,

$$-\nabla\,(P + (\frac{U^2}{2})) = \frac{1}{Re}\nabla \times (\nabla \times \vec{V}) - \vec{V} \times (\nabla \times \vec{V}) + \frac{\partial \vec{V}}{\partial t} \tag{2-12}$$

As previously discussed, all the quantities in the equations have been nondimensionalized. Dimensionless cylindrical coordinates x, r, and θ are used with $\vec{e}_x$, $\vec{e}_r$ and $\vec{e}_\theta$ denoting the unit vectors in the three coordinate directions, respectively. It is well known that the velocity profile for fully developed, three dimensional, incompressible flow in circular pipes can be expressed analytically by,

$$U = 2\ (1 - r^2) \tag{2-13}$$

This function has the shape of a paraboloid of revolution as shown in Figure 2-1.

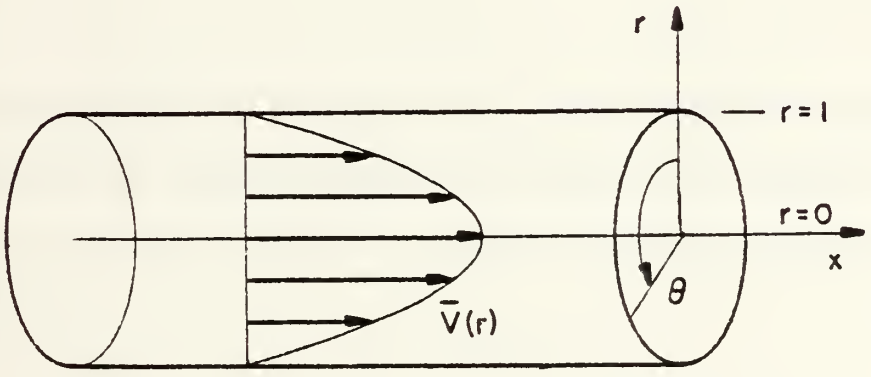


Figure 2-1. Velocity Profile of Fluid Flow in a Pipe

The nondimensional mean velocity vector $\vec{V}(r)$ of the flow is given by,

$$\vec{V}(r) = U\ \vec{e}_x = 2\ (1 - r^2)\ \vec{e}_x \tag{2-14}$$

A second vector quantity $\vec{\Omega}$, the mean vorticity, is defined as the curl of the mean velocity vector and is given by,

$$\vec{\Omega} = \nabla \times \vec{V} = 4\ r\ \vec{e}_{\theta} \tag{2-15}$$

which will be used later.

B. THE VORTICTY TRANSPORT EQUATION

The vorticity transport equation is now obtained by taking the curl of the Navier-Stokes equation (2-12), resulting in,

$$- \nabla \times \nabla \left(P + \left(\frac{U^2}{2}\right)\right) = \frac{1}{Re} \nabla \times (\nabla \times (\nabla \times \vec{V})) - \nabla \times (\vec{V} \times (\nabla \times \vec{V})) + \frac{\partial(\nabla \times \vec{V})}{\partial t} \tag{2-16}$$

The primary reason for taking the curl of the Navier-Stokes equation is to eliminate the unknown scalar on the left side of equation (2-16). Since the curl of the gradient of a scalar is equal to zero, equation (2-6) reduces to,

$$\frac{1}{Re} \nabla \times (\nabla \times (\nabla \times \vec{V})) - \nabla \times (\vec{V} \times (\nabla \times \vec{V})) + \frac{\partial(\nabla \times \vec{V})}{\partial t} = 0 \tag{2-17}$$

Equation (2-17) can be simplified by substituting the mean vorticity vector relation equation (2-15) into equation (2-17) resulting in,

$$\frac{1}{Re} \nabla \times (\nabla \times \vec{\Omega}) - \nabla \times (\vec{V} \times \vec{\Omega}) + \frac{\partial \vec{\Omega}}{\partial t} = 0 \tag{2-18}$$

In order to analyze this flow field and in particular to determine the transition from laminar to turbulent flow, a disturbance of small amplitude is introduced into the flow. It is assumed that the resulting flow is made up of the steady state, laminar flow and a small perturbation flow superimposed on the laminar flow. The total velocity and vorticity respectively become,

$$\vec{V}' = \vec{V} + \vec{v} \tag{2-19}$$

and

$$\vec{\Omega}' = \vec{\Omega} + \vec{\omega} \tag{2-20}$$

where $\vec{v}$ and $\vec{\omega}$ are the velocity perturbations and vorticity perturbations, respectively. It is of course a necessary requirement that the continuity equation for the total flow be satisfied here as well,

$$\nabla \cdot \vec{V}' = \nabla \cdot \vec{V} + \nabla \cdot \vec{v} = 0 \tag{2-21}$$

In view of this relation and of equation (2-2), the perturbation flow independently satisfies the continuity condition and

$$\nabla \cdot \vec{v} = 0 \tag{2-22}$$

The introduction of the perturbation quantities into the vorticity transport equation is accomplished by substituting $\vec{V}'$ for $\vec{V}$ and $\vec{\Omega}'$ for $\vec{\Omega}$ into equation (2-18). This results in,

$$\frac{1}{Re} \nabla \times (\nabla \times (\vec{\Omega} + \vec{\omega})) - \nabla \times [(\vec{V} + \vec{v}) \times (\vec{\Omega} + \vec{\omega})] + \frac{\partial(\vec{\Omega} + \vec{\omega})}{\partial t} = 0 \tag{2-23}$$

which is the equation for the total or resultant flow field. Upon expanding equation (2-23), applying the mean steady flow assumption, $\partial\vec{V}/\partial t = 0$, and subtracting equation (2-18) from it, equation (2-23) becomes,

$$\frac{1}{Re}\nabla\times(\nabla\times\vec{\omega}) - \nabla\times[(\vec{V}\times\vec{\omega}) - (\vec{\Omega}\times\vec{v}) + (\vec{v}\times\vec{\omega})] + \frac{\partial\vec{\omega}}{\partial t} = 0 \tag{2-24}$$

Now equation (2-23) is linear in the perturbation quantities except for the second order term, $(\vec{v}\times\vec{\omega})$. Since the perturbations are assumed to be small, this term can be neglected.

The perturbation vorticity is defined as,

$$\vec{\omega} = \nabla\times\vec{v} \tag{2-25}$$

which is analogous to equation (2-15) for the mean flow. Making this substitution in equation (2-24) yields the linearized vorticity transport equation,

$$\frac{1}{Re}\nabla\times(\nabla\times(\nabla\times\vec{v})) - \nabla\times(\vec{V}\times(\nabla\times\vec{v})) + \nabla\times(\nabla\times(\vec{V}\times\vec{v})) + \nabla\times\frac{\partial\vec{v}}{\partial t} = 0 \tag{2-26}$$

As previously mentioned, the continuity equation (2-22) for the perturbation velocity must be satisfied. A convenient way to insure this is to express $\vec{v}$ in terms of a velocity vector potential function $\vec{W}$ as follows,

$$\vec{v} = \nabla \times \vec{W} \tag{2-27}$$

It can be verified that if $\vec{v}$ is defined in this way, equation (2-22) is satisfied identically for any arbitrary vector function $\vec{W}$.

Instead of proceeding directly with the real vector functions $\vec{v}$ and $\vec{W}$, it is advantageous to work with the Fourier transforms of $\vec{v}$ and $\vec{W}$, which are in general complex functions. This is the form which is used here. It can be shown, because of the linearity of the basic equation, that if a solution can be found for the Fourier transforms, then the corresponding real vector functions will satisfy the basic equation as well. A detailed explanation of these functions was presented more elegantly by Gawain [Ref. 13]. The vector potential function $\vec{W}$, now representing the Fourier transform, can be expressed as,

$$\vec{W} = [F(r)\ \vec{e}_x + G(r)\ \vec{e}_r + H(r)\ \vec{e}_\theta]\ e^X \tag{2-28}$$

where X is a convenient abbreviation for the complex quantity,

$$X = i\alpha x + in\theta + \gamma t \tag{2-29}$$

The axial wave number α is a real quantity greater than or equal to zero which in the exponential form represents purely sinusoidal axial variations of $\vec{v}$. The angular wave number n is a positive integer and represents sinusoidal angular variations of $\vec{v}$ which are periodic with respect to coordinate Θ. The complex frequency γ contained in the exponent is of the form,

$$e^{\gamma t} = e^{(\gamma_R + i\gamma_I)t} = e^{\gamma_R t} (\cos \gamma_I t + i \sin \gamma_I t) \qquad (2\text{-}30)$$

The quantity γ_R clearly represents the exponential time rate of growth or decay of the perturbation amplitude. The basic criteria for hydrodynamic stability can be defined in terms of γ_R. Positive values of γ_R represent growth and signify flow instability while negative values of γ_R represent decay and signify flow stability.

The vorticity transport equation can now be expressed in terms of the velocity vector, $\vec{V}$ and the complex vector potential, $\vec{W}$. By making the substitution for $\vec{v}$, equation (2-26) now becomes,

$$\frac{1}{Re} \nabla \times (\nabla \times (\nabla \times (\nabla \times \vec{W}))) - \nabla \times (\vec{V} \times (\nabla \times (\nabla \times \vec{W})))$$

$$+ \nabla \times (\nabla \times (\vec{V} \times (\nabla \times \vec{W}))) + \nabla \times (\nabla \times \frac{\partial \vec{W}}{\partial t}) = 0 \qquad (2\text{-}31)$$

In convenient shorthand notation, equation (2-31) becomes,

$$\text{Equation (2-31)} = \vec{\Gamma}(r)e^{X} = [\Gamma_x(r)\, \vec{e}_r + \Gamma_r(r)\, \vec{e}_\theta + \Gamma_\theta(r)\, \vec{e}\,]\, e^{X} = 0 \qquad (2\text{-}32)$$

and can be expressed in matrix notation as,

$$\vec{\Gamma} = \begin{Bmatrix} \Gamma_x \\ \Gamma_r \\ \Gamma_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}33)$$

Therefore, the three components of the vorticity transport equation are equivalent to three simultaneous scalar equations, namely,

$$\Gamma_x = 0$$
$$\Gamma_r = 0 \qquad (2\text{-}34)$$
$$\Gamma_\theta = 0$$

Of these three equations, only two are actually independent since, as can be seen from equation (2-31), $\vec{\Gamma}\, e^x$ is itself the curl of another vector. Therefore $\vec{\Gamma}\, e^x$ must necessarily be nondivergent. It can be shown that the components of $\vec{\Gamma}$ satisfy the identity, $\nabla \cdot (\vec{\Gamma}\, e^x) \equiv 0$, or,

$$i\alpha\Gamma_x + \frac{1}{r} D\,(r\Gamma_r) + \left(\frac{in}{r}\right)\Gamma_\theta \equiv 0 \qquad (2\text{-}35)$$

where D is the symbol for the differential operator d/dr. Two linearly independent equations can now be formulated from equations (2-34) given the constraint of equation (2-35). This can be reduced to some appropriate linear combination of two of the equations which is set equal to zero and to the third equation which is also set equal to zero. This results in a system of two coupled, fourth order differential equations in three unknowns. Recall that the vector potential function $\vec{W}$ is expressed in terms of F(r), G(r), and H(r). It can be shown that one of these variables is redundant and may be eliminated without any loss of generality in the solution of the equations. For the present case, the solution assumes its most convenient form if,

$$F(r) = 0 \qquad (2\text{-}36)$$

This convention has therefore been adopted and now G(r) and H(r) become the two coupled eigenfunctions of the governing fourth order differential equations. Returning to the problem of selecting the proper linear combination of equations comprising $\vec{\Gamma}$, it is important to choose this combination in an optimum fashion by analyzing the algebraic structure of these equations. Table 2-1 is taken from reference [13], corrected, and duplicated here for that purpose. Notice that equations (1) and (3) in the table are third order in G. If the particular linear combination of these equations is chosen as appears in equation (4), the result reduces to an equation of second order in G. This does not affect the order of the result since the function H remains fourth order.

Table 2-1. Properties of the Vorticity Transport Equations

EQUATION	TERM OF HIGHEST ORDER IN G	TERM OF HIGHEST ORDER IN H	HIGHEST NEGATIVE POWER OF r
(1) $\Gamma_x(r) = 0$	$\frac{i\alpha}{Re} D^3G$	$-\frac{n\alpha}{rRe} D^2H$	r^{-3}
(2) $\Gamma_r(r) = 0$	$\frac{1}{Re}(\alpha^2 + \frac{n^2}{r^2}) D^2G$	$\frac{in}{rRe} D^3H$	r^{-4}
(3) $\Gamma_\theta(r) = 0$	$\frac{in}{rRe} D^3G$	$-\frac{1}{Re} D^4H$	r^{-4}
(4) $-\frac{n}{\alpha r}\Gamma_x(r)+\Gamma_\theta(r)=0$	$-\frac{4\,in}{r^2Re} D^2G$	$-\frac{1}{Re} D^4H$	r^{-4}

The formulation of the problem is simplified greatly by choosing two independent vorticity transport equations in the form,

$$\Gamma_r = 0 \tag{2-37}$$

and

$$-\frac{n}{\alpha r}\Gamma_x + \Gamma_\theta = 0 \tag{2-38}$$

C. THE VORTICITY TRANSPORT MATRIX EQUATION

While the fourth order linear vector operations and the associated algebra becomes very lengthy and rather tedious, the investigator attempting to expand the final form of the linearized vorticity transport equation (2-31) should pay sufficient attention to these details. Recall that the curl operation on a vector in cylindrical coordinates takes the form,

$$\nabla \times \vec{W} = \begin{vmatrix} \frac{1}{r}\vec{e}_x & \frac{1}{r}\vec{e}_r & \vec{e}_\theta \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ 0 & Ge^x & rHe^x \end{vmatrix} \tag{2-39}$$

After all the nabla (∇) operations are complete, the final equation will be in the form of a vector equation as defined in equation (2-32). The first vorticity transport equation consists of the coefficient represented by Γ_r and is set equal to zero. The second equation consists of

the previously defined linear combination of the coefficients represented by Γ_x and Γ_θ and is also set equal to zero. The two resulting simultaneous ordinary differential equations are then expanded and the coefficients of the derivatives regrouped so that equations (2-37) and (2-38) can be conveniently expressed in matrix format as follows,

$$[M_4]\begin{Bmatrix} D^4G \\ D^4H \end{Bmatrix} + [M_3]\begin{Bmatrix} D^3G \\ D^3H \end{Bmatrix} + [M_2]\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} + [M_1]\begin{Bmatrix} DG \\ DH \end{Bmatrix}$$

$$+ [M_0]\begin{Bmatrix} G \\ H \end{Bmatrix} = \gamma\left([N_2]\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} + [N_1]\begin{Bmatrix} DG \\ DH \end{Bmatrix} + [N_0]\begin{Bmatrix} G \\ H \end{Bmatrix}\right) \qquad (2\text{-}40)$$

As it turns out, the equations are a pair of coupled homogeneous differential equations that can be solved as an eigenvalue problem. This technique is discussed in the chapter on Numerical Methods.

The matrices which appear in equation (2-40) are 2 X 2 matrices and are summarized in detail in Appendix A. Equation (2-40) and the respective matrix elements contained in Appendix A are the generalized vorticity transport equations for all angular wave numbers n, at all axial wave numbers α, and all Reynolds numbers. As will be explained in detail in the chapter on Boundary Conditions, the generalized equations are transformed by appropriate changes of variable depending on the angular wave number. The changes of variables from the eigenfunctions G(r) and H(r) to the new eigenfunctions P(r) and Q(r), respectively, are determined by the original boundary conditions that must be satisfied, particularly at the pipe axis.

D. EQUATIONS FOR n = 0

For the case n = 0, equation (2-40) is greatly simplified in that the two equations uncouple and allow an independent investigation of either eigenfunction, G(r) or H(r). Miscellaneous trial calculations made earlier suggest that the solutions for G(r) are probably stable for all values of α and Re. Therefore, only the solutions for H(r) were investigated. It should also be noted that the examination of the axial perturbation velocity is of particular interest in this analysis and, in the case for n = 0, is not a function of G(r). The change of variable required to satisfy the boundary conditions is,

$$H(r) = r\,Q(r) \tag{2-41}$$

Taking the derivatives of H(r) yields,

$$\begin{aligned} DH &= r\,DQ + Q \\ D^2H &= r\,D^2Q + 2\,DQ \\ D^3H &= r\,D^3Q + 3\,D^2Q \\ D^4H &= r\,D^4Q + 4\,D^3Q \end{aligned} \tag{2-42}$$

With the substitution of equations (2-41) and (2-42) into equation (2-40), the following expression is obtained,

$$M_4' D^4Q + M_3' D^3Q + M_2' D^2Q + M_1' DQ + M_0' Q = \gamma (N_2' D^2Q + N_1' DQ + N_0' Q) \tag{2-43}$$

The primed coefficients are actually the elements from row 2 and column 2 of the newly transformed coefficient matrices and are not enclosed in matrix brackets. However, in general, the new coefficient matrices $[M_i']$ and $[N_j']$, (where the indices i = 0, 1, 2, 3, 4 and j = 0, 1, 2), are formed after the change of variable is made and the terms are collected. The primed coefficients used to solve the transformed vorticity transport equation in terms of Q(r) for n = 0 are defined in Appendix B, Part A.

E. EQUATIONS for n = 1

For the case n = 1, eigenfunctions G(r) and H(r) do not become uncoupled and must be solved for in a system of simultaneous equations in a form similar to equation (2-40). An additional parameter H(0) is introduced into the equations by the change of variables. This particular change of variables is required because of the complicated nature of the boundary conditions at the axis and is thoroughly explained in the chapter on Boundary Conditions. For n = 1, the change of variables required to satisfy the boundary conditions are,

$$\begin{aligned} G(r) &= -i\, H(0) + r^2\, P(r) \\ H(r) &= H(0) + r^2\, Q(r) \end{aligned} \qquad (2\text{-}44)$$

Taking the derivatives of G(r) yields,

$$\begin{aligned} DG &= r^2\, DP + 2r\, P \\ D^2G &= r^2\, D^2P + 4r\, DP + 2\, P \end{aligned} \qquad (2\text{-}45)$$

and taking the derivatives of H(r) yields,

$$\begin{aligned} DH &= r^2\, DQ + 2r\, Q \\ D^2H &= r^2\, D^2Q + 4r\, DQ + 2\, Q \\ D^3H &= r^2\, D^3Q + 6r\, D^2Q + 6\, DQ \\ D^4H &= r^2\, D^4Q + 8r\, D^3Q + 12\, D^2Q \end{aligned} \tag{2-46}$$

Remember that in general $D(H(0)) \neq DH(0)$, where $DH(0)$ is the value of the first derivative of H(r) evaluated at the point r = 0. Since $D(H(0)) = 0$ here, it vanishes from the expressions for the derivatives of G(r) and H(r). In order to accommodate H(0) in the system of equations where it appears explicitly, two additional 2 X 1 column matrices are required, namely $[M_5']$ and $[N_3']$. After the changes of variable have been made, equation (2-40) now becomes,

$$[M_4'] \begin{Bmatrix} D^4P \\ D^4Q \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3P \\ D^3Q \end{Bmatrix} + [M_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [M_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [M_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix}$$

$$+ [M_5'] \{H(0)\} = \gamma \left([N_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [N_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [N_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix} + [N_3'] \{H(0)\} \right) \tag{2-47}$$

The coefficient matrices of equation (2-47) used to solve the transformed vorticity transport equations in terms of P(r) and Q(r) for n = 1 are defined in Appendix B, Part B.

F. EQUATIONS FOR n = 6

For the case n = 6, eigenfunctions G(r) and H(r) remain coupled. The change of variables does not introduce any additional parameters and the system of simultaneous equations is solved in the form of equation (2-40) once again. The change of variables required to satisfy the boundary conditions for n = 6 and additionally for all angular wave numbers n > 6 are,

$$G(r) = r^4 P(r)$$
$$H(r) = r^3 Q(r) \qquad (2\text{-}48)$$

Taking the derivatives of G(r) yields,

$$DG = r^4 DP + 4r^3 P$$
$$D^2G = r^4 D^2P + 8r^3 DP + 12r^2 P \qquad (2\text{-}29)$$

and taking the derivatives of H(r) yields,

$$DH = r^3 DQ + 3r^2 Q$$
$$D^2H = r^3 D^2Q + 6r^2 DQ + 6r Q$$
$$D^3H = r^3 D^3Q + 9r^2 D^2Q + 18r DQ + 6 Q$$
$$D^4H = r^3 D^4Q + 12r^2 D^3Q + 36r D^2Q + 24 DQ \qquad (2\text{-}50)$$

Substituting equations (2-48), (2-49), and (2-50) into equation (2-40) results in the transformed coefficient matrices for the system of equations. The coefficient matrices of the functions P(r) and Q(r) and their respective derivatives for n = 6 are defined in Appendix B, Part C.

III. BOUNDARY CONDITIONS

The incorrect formulation of the boundary conditions for the linearized vorticity transport equations at the pipe axis has probably been the primary reason that previous investigators have failed to predict the onset of flow instabilities in circular pipes at any Reynolds number. Gawain [Ref. 13] suggested that if these boundary conditions are formulated in a rigorous and systematic fashion and applied to the vorticity transport equations, a correct numerical solution may be computed which agrees with experimental results. It can be seen from equations (2-40) and from the coefficient matrices of Appendix A that the two vorticity transport equations are coupled, except for the case n = 0. The resulting pair of differential equations are second order in G(r) and fourth order in H(r). In order to obtain a determinate solution of the equations, a total of six boundary conditions are required. More specifically, at least two of the boundary conditions must involve G(r) or its derivatives and at least four of the boundary conditions must involve H(r) or its derivatives. It might be expected that when the equations are coupled, the boundary conditions may also be coupled. That is to say that the boundary conditions may consist of various linear combinations of the functions and their derivatives, which turns out to be the case and is shown later.

The boundary conditions at the pipe wall are determined in a fairly straight forward manner. They are derived from the fact that the components of the perturbation velocity must vanish at the wall. From the definition of the perturbation velocity, defined previously as,

$$\vec{v} = \nabla \times \vec{W} \tag{2-27}$$

it can be shown that the following three boundary conditions result.

$$\begin{aligned} G(1) &= 0 \\ H(1) &= 0 \\ DH(1) &= 0 \end{aligned} \tag{3-1}$$

Since six boundary conditions are required and three have been determined at the pipe wall, there remain three boundary conditions to be determined at the pipe axis. Of these three conditions, at least one must involve G(r) or its derivatives and at least two must involve H(r) or its derivatives.

The proper derivation of the three boundary conditions at the axis turns out to be a non-trival problem. As can be seen in Table 2-1, the highest negative power of r that appears in the equations is r^{-4}. Closer inspection of the vorticity transport equations of equation (2-40) and the coefficient matrices in Appendix A reveals that the equations contain terms in r^{-4}, r^{-3}, r^{-2}, r^{-1}, and r^{0}. At first glance, it may appear that these equations are not satisfied in the limit as r approaches zero. It should be noted that it would be incorrect to multiply the equations through by any positive power of r since that would alter the degree of the singularity of the equations at the axis.

This problem can be resolved and subsequently the boundary conditions at the axis can be deduced rigorously by expanding the functions G(r) and H(r) as power series in r. The highest negative power of r is r^{-4} and appears in the coefficient matrix $[M_0]$ for the pair of functions G(r) and H(r). The series expansion for G(r), and similarly for H(r), is carried to the fourth derivative as follows,

$$G(r) = G(0) + DG(0)r + D^2G(0)\frac{r^2}{2!} + D^3G(0)\frac{r^3}{3!} + D^4G(0)\frac{r^4}{4!} + \ldots\ldots \quad (3\text{-}2)$$

It can be seen that when the series expansion expressions for G(r) and H(r) are substituted into equation (2-40), the higher order terms of equation (3-2) that contain powers of r greater than four will vanish in the limit as r approaches zero. For the first derivatives of G(r) and H(r), the highest negative power of r that appears in the coefficient matrix $[M_1]$ is r^{-3}. Therefore, the series expansion for the first derivative of G(r), and similarly for H(r), up to the fourth derivative term is,

$$DG(r) = 0 + DG(0) + D^2G(0)r + D^3G(0)\frac{r^2}{2!} + D^4G(0)\frac{r^3}{3!} + \ldots\ldots \quad (3\text{-}3)$$

The remaining derivatives of the functions are expanded in a similar manner through the fourth derivative term. The resulting power series approximations for G(r) and H(r) and their derivatives are then substituted into equation (2-40). The coefficients are then regrouped in ascending powers of r.

As mentioned previously, the equations must be satisfied in the limit as r approaches zero. Therefore, the coefficients of the functions that contain r^{-4}, r^{-3}, r^{-2}, r^{-1}, and r^{0} must all be zero. The remaining coefficients will contain positive powers of r and will vanish in the limit as r approaches zero. By considering only the terms containing negative powers of r and setting those coefficients equal to zero, five pair of equations in ten unknowns remain so that the boundary conditions can be determined. The unknowns are the functions of G(r) and H(r) and their first four derivatives evaluated at the pipe axis, r = 0. It is convenient to summarize these equations in matrix format below,

$$\frac{(n^2-1)}{Re}\,[C_1]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-4}$$

$$\frac{n^2}{Re}\,[C_2]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-5}$$

$$\frac{(n^2-1)}{Re}\,[C_3]\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + (i\alpha[C_4] - \gamma\,[D_4])\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-6}$$

$$\frac{n(n^2-4)}{Re}\,[C_5]\begin{Bmatrix} D^3G(0) \\ D^3H(0) \end{Bmatrix} + n\left(2i\alpha\left(1 - \frac{i\alpha}{Re}\right) + \gamma\right)[C_6]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-7}$$

$$\frac{(n^2-9)}{Re}\,[C_7]\begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + (i\alpha\,[C_8] - \gamma\,[D_8])\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix}$$

$$+ (i\alpha\,[C_9] - \gamma\alpha^2\,[D_9])\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3\text{-}8)$$

The 2 X 2 matrices which appear in equations (3-4) through (3-8) are defined in detail in Appendix C. Special conditions at the axis arise when the determinants of these matrices are evaluated at specific angular wave numbers. They are summarized in Appendix D.

For each specific value of angular wave number n, it is possible to derive a set of boundary conditions at the axis. Special conditions at the axis occur for angular wave numbers that cause the coefficients in equations (3-4) through (3-8) to equal zero or the determinants in Appendix D to be zero. That is to say, if a coefficient vanishes or the pair of equations are linearly dependent, the functions may be arbitrarily specified and are in general not equal to zero. But if the coefficient, appearing in equations (3-4) through (3-8) is non-zero and the determinant exists, the pair of functions must be identically zero. This situation occurs for all five pairs of variables when the angular wave number $n \geqq 6$.

By taking these special conditions into account for the case n = 0, it can be deduced from equations (3-6) and (3-8) that,

$$\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3\text{-}9)$$

Carrying out this procedure for n = 1, 2, 3, 4, 5, and 6 yields a simplified set of boundary conditions at the axis. The following sets of equations (3-10) through (3-16) are essentially the same as those of Gawain [Ref. 13] but were independently checked and appear here with a few minor corrections.

$$\underline{n = 0}$$

$$\begin{array}{ll} G(0) = 0 & H(0) = 0 \quad \text{(a)} \\ D^2G(0) = 0 & D^2H(0) = 0 \quad \text{(b)} \end{array} \tag{3-10}$$

$$\underline{n = 1}$$

$$\begin{array}{ll} G(0) + i\,H(0) = 0 & \text{(a)} \\ DG(0) = 0 \qquad DH(0) = 0 & \\ D^3G(0) = 0 \qquad D^3H(0) = 0 & \end{array} \tag{3-11}$$

$$-\frac{8}{Re}\,[C_7]\begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + (i\alpha\,[C_8] - \gamma\,[D_8])\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} \quad \text{(b)}$$

$$+ (i\alpha\,[C_9] - \gamma\,\alpha^2\,[D_9])\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{n = 2}$$

$$\begin{array}{ll} G(0) = 0 \qquad H(0) = 0 & \\ DG(0) + i\,DH(0) = 0 & \text{(a)} \\ D^2G(0) = 0 \qquad D^2H(0) = 0 & \\ D^4G(0) = 0 \qquad D^4H(0) = 0 & \text{(b)} \end{array} \tag{3-12}$$

$n = 3$

$$G(0) = 0 \qquad H(0) = 0$$

$$DG(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^2G(0) + i\,D^2H(0) = 0 \qquad \text{(b)} \qquad \text{(3-13)}$$

$$D^3G(0) = 0 \qquad D^3H(0) = 0$$

$n = 4$

$$G(0) = 0 \qquad H(0) = 0$$

$$DG(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^2G(0) = 0 \qquad D^2H(0) = 0$$

$$D^3G(0) + i\,D^3H(0) = 0 \qquad \text{(b)} \qquad \text{(3-14)}$$

$$D^4G(0) = 0 \qquad D^4H(0) = 0$$

$n = 5$

$$G(0) = 0 \qquad H(0) = 0$$

$$DG(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^2G(0) = 0 \qquad D^2H(0) = 0$$

$$D^3G(0) = 0 \qquad D^3H(0) = 0 \qquad \text{(b)} \qquad \text{(3-15)}$$

$$D^4G(0) + i\,D^4H(0) = 0$$

$n \geq 6$

$$G(0) = 0$$

$$DG(0) = 0 \qquad H(0) = 0$$

$$D^2G(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^3G(0) = 0 \qquad D^2H(0) = 0$$

$$D^4G(0) = 0 \qquad D^3H(0) = 0 \qquad \text{(b)} \qquad \text{(3-16)}$$

$$D^4H(0) = 0$$

Recall that exactly three boundary conditions are required at the axis to finalize the solution of the vorticity transport equations. Equations (3-10) through (3-16) contain from four to ten boundary constraints including linear combinations of several boundary conditions, as suggested earlier. By introducing appropriate changes of variables, the pair of functions in G(r) and H(r) and their respective derivatives can be transformed to a new pair of functions in terms of P(r) and Q(r) and their derivatives. The transformed system of equations take the form of equation (2-40) for all values of n, except for n = 1 where equation (2-47) is applicable. The change of variables is dependent on each specific value of n up through six and is chosen in such a way that the boundary conditions in subset (a) of equation (3-10) through (3-16) are satisfied identically. The boundary conditions in subset (b) must be satisfied explicitly in the finite difference equations near the axis. Development of the finite difference equations is discussed in the chapter on Numerical Methods.

The following sets of equations are taken from Gawain [Ref. 13] and are included here so that a complete treatment of the subject on boundary conditions is contained in this paper.

$n = 0$

Change of Variables

$$G(r) = rP(r)$$
$$H(r) = rQ(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-17)

$$DP(0) = 0 \qquad DQ(0) = 0$$
$$D^3Q(0) = 0 \qquad \text{(b)}$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$
$$DQ(1) = 0 \qquad \text{(c)}$$

The solutions for P(r) and Q(r) become uncoupled for n = 0

$n = 1$

Change of Variables

$$G(r) = -i\,H(0) + r^2\,P(r)$$
$$H(r) = H(0) + r^2\,Q(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-18)

$$DP(0) = 0 \qquad DQ(0) = 0$$

$$-\frac{96}{Re}[C_7]\begin{Bmatrix} D^2P(0) \\ D^2Q(0) \end{Bmatrix} + 2i\alpha[C_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + i\alpha[C_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)$$

$$= +\gamma\left(2[D_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + \alpha^2\,[D_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)\right) \qquad \text{(b)}$$

Boundary Conditions at Wall

$$-i\,H(0) + P(1) = 0 \qquad H(0) + Q(1) = 0$$
$$-2H(0) + DQ(1) = 0 \qquad \text{(c)}$$

---

n = 2

Change of Variables

$$G(r) = -\,i\,r\,DH(0) + r^2\,P(r)$$

$$H(r) = r\,DH(0) + r^2\,Q(r) \qquad (a)$$

Boundary Conditions at Axis (3-19)

$$P(0) = 0 \qquad Q(0) = 0$$

$$D^2P(0) = 0 \qquad D^2Q(0) = 0 \qquad (b)$$

Boundary Conditions at Wall

$$-\,i\,DH(0) + P(1) = 0 \qquad DH(0) + Q(1) = 0$$

$$-DH(0) + DQ(1) = 0 \qquad (c)$$

---

n = 3

Change of Variables

$$G(r) = r^2\,P(r)$$

$$H(r) = r^2\,Q(r) \qquad (a)$$

Boundary Conditions at Axis (3-20)

$$P(0) + i\,Q(0) = 0 \qquad (b)$$

$$DP(0) = 0 \qquad DQ(0) = 0$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$

$$DQ(1) = 0 \qquad (c)$$

---

n = 4

Change of Variables

$$G(r) = r^3 P(r)$$

(a)

$$H(r) = r^3 Q(r)$$

Boundary Conditions at Axis (3-21)

$$P(0) + i\, Q(0) = 0$$

(b)

$$DP(0) = 0 \qquad DQ(0) = 0$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$

(c)

$$DQ(1) = 0$$

---

n = 5

Change of Variables

$$G(r) = r^3 P(r)$$

(a)

$$H(r) = r^3 Q(r)$$

Boundary Conditions at Axis (3-22)

$$P(0) = 0 \qquad Q(0) = 0$$

(b)

$$DP(0) + i\, DQ(0) = 0$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$

(c)

$$DQ(1) = 0$$

$n \geq 6$

Change of Variables

$$G(r) = r^4 P(r)$$

$$H(r) = r^3 Q(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-23)

$$P(0) = 0 \qquad Q(0) = 0$$

$$DQ(0) = 0 \qquad \text{(b)}$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$

$$DQ(1) = 0 \qquad \text{(c)}$$

It should be noted that the change of variable relations in subset (a) of equations (3-17) through (3-23) does not change the order of the derivatives appearing in the transformed vorticity transport equations. Upon close examination of subsets (b) and (c) of these equations, it is apparent that the correct number of boundary conditions required for the solution of the transformed equations has resulted from the change of variables. The vorticity transport equations are now expressed in terms of P(r) and Q(r) and their derivatives and have exactly three boundary conditions at the pipe wall for all values of n and three boundary conditions at the axis for all values of n, except for n = 1 and 2. The fact that there are four boundary conditions at the axis for angular wave numbers n = 1 and 2 warrants further comment here.

Notice in equations (3-18a) and (3-19a) for the cases n = 1 and 2 that an additional parameter is introduced by the change of variable equations. The additional parameter for n = 1 is H(0) and for n = 2 is DH(0). The introduction of the parameter for each case requires the specification of four boundary conditions at the axis instead of the usual three. This conclusion was reached by Gawain [Ref. 13] and is analyzed in detail in his paper. There is one other peculiarity about the boundary conditions at the axis for n = 1 that should be mentioned. The axis boundary conditions in equation (3-18b) contain a pair of equations that involve the complex perturbation frequency γ, which represents an eigenvalue of the solution for the equations. The presence of γ in the axis boundary condition for n = 1 is an important factor in formulating a solution for the vorticity transport equations and is included in the eigensystem of the finite difference equations. The details of the implementation are discussed later.

The solution of the problem for predicting the onset of flow instabilities in circular pipes is now at hand. The original vorticity transport equations (2-40) are transformed to the functions P(r) and Q(r) and their respective derivatives by the appropriate change of variables depending on the specific angular wave number being investigated. After imposing the rigorously deduced boundary conditions, a solution can be determined in the perturbation quantities.

IV. PERTURBATION VELOCITY

The solution of the transformed vorticity transport equations developed in the previous chapter can now be expressed in terms of the perturbation quantities P(r) and Q(r). The original functions G(r) and H(r) are actually components of the vector potential function W defined in equation (2-28),

$$\vec{W} = [F(r)\,\vec{e}_x + G(r)\,\vec{e}_r + H(r)\,\vec{e}_\theta]\,e^{x} \qquad (2\text{-}28)$$

The axial component F(r) was previously set to zero since one function could be arbitrarily specified. From equation (2-27) the perturbation velocity vector was expressed as,

$$\vec{v} = \nabla \times \vec{W} \qquad (2\text{-}27)$$

Therefore the functions G(r) and H(r) represent the behavior of the perturbation velocity in the flow field. Referring back to the definition of the curl operation in equation (2-39), the relation for $\nabla \times \vec{W}$ has already been set up. The resulting expression for the perturbation velocity is given by,

$$\vec{v} = u\,\vec{e}_x + v\,\vec{e}_r + w\,\vec{e}_\theta \qquad (4\text{-}1)$$

The components of the perturbation velocity are,

$$u = DH(r) + \frac{H(r)}{r} - \frac{in}{r} G(r) \qquad (4\text{-}2)$$

$$v = -i\alpha H(r) \qquad (4\text{-}3)$$

$$w = i\alpha G(r) \qquad (4\text{-}4)$$

Digressing for a moment, it can be readily seen that since the perturbation velocity necessarily vanishes at the pipe wall, r = 1, the boundary conditions at the wall in equation (3-1) can be deduced.

Recall that the functions P(r) and Q(r) appear as the result of the change of variables for each specific angular wave number contained in subset (a) of equations (3-17) through (3-23). Analysis of the axial component of the perturbation velocity u becomes useful in interpreting the behavior of the functions P(r) and Q(r) along the radius of the pipe. Equations for the axial perturbation velocity in terms of P(r) and Q(r) are derived from equations (4-2) through (4-4) with the change of variable applied for each angular wave number being investigated and appear as follows,

$$n = 0 : \quad u = 2\,Q + r\,DQ \qquad (4\text{-}5)$$

$$n = 1 : \quad u = 3\,r\,Q + r^2 DQ - i\,r\,P \qquad (4\text{-}6)$$

$$n = 6 : \quad u = 4\,r^2\,Q + r^3 DQ - i\,6\,r^3\,P \qquad (4\text{-}7)$$

The axial perturbation velocity is computed in part VII of the main investigative program for each angular wave number. To prepare the perturbation velocity u for plotting versus pipe radius, it is first

normalized. Since this velocity is in general a complex quantity, the perturbation velocity with the largest magnitude is determined by,

$$|u_c|_{MAX} = (u_R^2 + u_I^2)^{\frac{1}{2}} \tag{4-8}$$

The normalizing velocity constant will then become u_c and the maximum normalized velocity will become,

$$\frac{u}{u_c} = 1 + i(0) \tag{4-9}$$

The remaining normalized velocities are found by dividing through by u_c. The normalized perturbation velocity is then plotted versus the normalized pipe radius for various axial wave numbers and Reynolds numbers at the three angular wave numbers investigated. The plots and results are discussed in detail in the chapter on Results.

V. NUMERICAL METHODS

A. GENERAL METHODS USED

In general, the vorticity transport equations are a pair of coupled homogeneous differential equations which are second order in the perturbation quantity G(r) and fourth order in the perturbation quantity H(r). A change of variables is introduced so that the number of boundary conditions at the axis is reduced to three for each angular wave number investigated, except for n = 1. Recall for this case that there are four boundary conditions imposed at the axis because of the introduction of the additional parameter H(0). By applying the change of variables to the original pair of equations, the vorticity transport equations are transformed into an equivalent pair of coupled homogeneous differential equations that are second order in P(r) and fourth order in Q(r). The solution of the vorticity transport equations will now be in terms of the perturbation quantities P(r) and Q(r) which are also referred to as eigenfunctions, and γ which is an eigenvalue. The axial perturbation velocity is expressed in terms of P(r) and Q(r) as derived previously and is examined to determine the behavior of the perturbation quantities. The resulting eigenvalue γ is of primary interest since the magnitude and sign of the real part of γ determines the relative stability or instability of the pipe flow problem.

A numerical solution of the transformed vorticity transport equations is now sought. The convenient matrix format of the equations is given below.

$$[M_4'] \begin{Bmatrix} D^4P \\ D^4Q \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3P \\ D^3Q \end{Bmatrix} + [M_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [M_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [M_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix} =$$

$$\gamma\, [N_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [N_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [N_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix} \qquad (5\text{-}1)$$

The details of the transformed coefficient matrices for n = 0, 1, and 6 are given in Appendix B. It is convenient to approximate the derivatives of P(r) and Q(r) by a finite number of discrete unknowns along a radial mesh consisting of N interior points. It is not possible to compute the continuous values of the functions and their derivatives along the pipe radius using a digital computer. Therefore, discrete finite difference techniques are used. For the problem at hand, the normalized nondimensional radius of the pipe, which becomes the independent variable, is divided into a computational mesh. The standard method of representing this mesh is with uniformly spaced stations over the interval $0 < r < 1$, Figure 5-1.

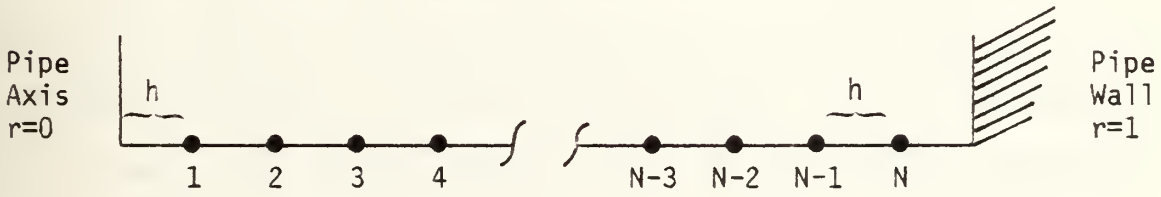


Figure 5-1. Radial Mesh, Standard Method

The vorticity transport equation coefficients are then evaluated at each station of the radial mesh resulting in N uniformily spaced discrete values with N + 1 equal intervals and N + 2 total points, including r = 0 and r = 1. The spacing between the stations is denoted by h = 1/(N + 1).

A refinement of the method for generating a computational mesh is referred to as the half station method and is used in this numerical analysis. All the radial stations of Figure 5-1 are moved to mid-interval or one half station (h/2) toward the axis with an additional station appearing at one half station from the pipe wall, as shown in Figure 5-2.

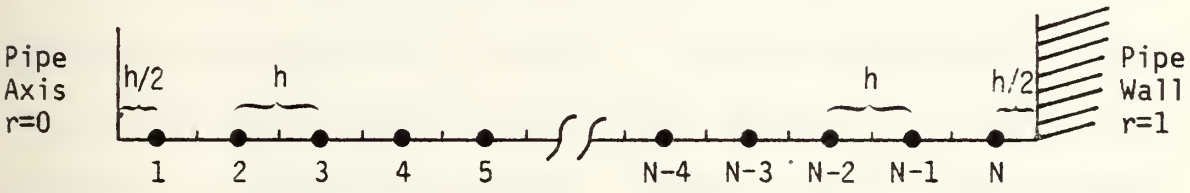


Figure 5-2. Radial Mesh, Half Station Method

Therefore for N interior stations, the method results in N intervals and N + 2 total points including the boundaries. The first station, r_1, is a distance h/2 from the axis and the Nth station, r_N, is a distance h/2 from the wall. The general expression for the ith station, corresponding to the ith increment of the radius, is given by,

$$r_i = h\,(i - \tfrac{1}{2})\ ;\ i = 1, 2, 3....N \qquad (5\text{-}2)$$

where h = 1/N.

An advantage of the half station method over the standard method is that there is better resolution at the stations near the boundaries where instabilities are thought to originate and where boundary layer effects are a factor. Therefore, the behavior of the perturbation quantities can be examined more closely. Additionally, better approximations for the derivatives of the functions P(r) and Q(r) can be realized with a smaller incremental distance to the points immediately adjacent to the boundaries. Even better resolution at the boundaries can be achieved by using a nonuniform computational mesh. Arnold [Ref. 11] introduced a change of independent variable by means of hyperbolic functions which included a mesh offset parameter so that he could control the concentration of mesh points at either the axis or wall. The method would certainly be useful in reducing computational time by reducing the total number of mesh points required for a satisfactory solution of the governing equations. However, the full implications of the nonuniform mesh with respect to α and γ dependence on the mesh offset parameter have not been fully explored. While this method is not used here, it may well prove to be useful in follow-on numerical investigations of the pipe flow problem.

The solution for the vorticity transport equations in terms of the perturbation quantities P(r) and Q(r) are now represented by a finite number of discrete unknowns. The governing differential equations are rewritten in finite difference form for N stations along the mesh resulting in a finite number of unknowns namely, P_1, P_2, $P_3 \ldots P_N$ and Q_1, Q_2, $Q_3 \ldots Q_N$. In general, this produces 2N simultaneous equations in 2N

unknowns. The details of the derivation of the finite difference equations are explained shortly. Substituting the finite difference equations into the vorticity transport equation matrix format of equation (5-1) will result in N pairs of coupled simultaneous equations. These equations are regrouped and subsequently solved in the following eigenvalue problem format,

$$[A]\ \{X\} = \gamma\ [B]\ \{X\} \tag{5-3}$$

The column vector $\{X\}$ represents the rearranged unknowns as

$$\{X\} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_i \\ \vdots \\ P_N \\ Q_1 \\ Q_2 \\ \vdots \\ Q_i \\ \vdots \\ Q_N \end{Bmatrix} \tag{5-4}$$

The symbol γ denotes the characteristic value or eigenvalue of the solution. Details of the composition of the [A] and [B] matrices are discussed for each specific angular wave number investigated.

B. FINITE DIFFERENCE EQUATIONS

The use of finite differencing techniques is the basis for solving the governing vorticity transport equations in the eigensystem equation (5-3). It is required that the governing equations must be satisfied at each of the interior stations i = 1, 2, 3....N, therefore producing 2N equations in 2N unknowns.

Except for stations adjacent to the boundaries, the derivatives of the perturbation quantities in the governing equations at the interior stations are usually approximated by standard central finite difference equations. These equations were taken from Table 7.27 of Ketter and Prawel [Ref. 14] and are summarized below,

$$DF_i = \frac{1}{12h}\left\{F_{i-2} - 8F_{i-1} + 8F_{i+1} - F_{i+2}\right\} + \frac{1}{30}\,h^4D^5F_i \qquad (5\text{-}5)$$

$$D^2F_i = \frac{1}{12h^2}\left\{-F_{i-2} + 16F_{i-1} - 30F_i + 16F_{i+1} - F_{i+2}\right\} + \frac{1}{90}\,h^4D^6F_i \qquad (5\text{-}6)$$

$$D^3F_i = \frac{1}{8h^3}\left\{F_{i-3} - 8F_{i-2} + 13F_{i-1} - 13F_{i+1} + 8F_{i+2} - F_{i+3}\right\} + \frac{7}{120}\,h^4D^7F_i \qquad (5\text{-}7)$$

$$D^4F_i = \frac{1}{6h^4}\left\{-F_{i-3} + 12F_{i-2} - 39F_{i-1} + 56F_i - 39F_{i+1} + 12F_{i+2} - F_{i+3}\right\} + \frac{7}{240}\,h^4D^8F_i \qquad (5\text{-}8)$$

where F is an arbitrary function.

As can be seen from the equations above, the derivative of a function at a particular station can be approximated by a linear combination of the discrete unknown functions about a central point. The term outside the brackets in each equation represents the truncation error of the corresponding finite difference approximation. Notice that the truncation error of equations (5-5) through (5-8) are all of order h^4 (Oh^4). It was hypothesized and shown by Gawain and Ball [Ref. 15] that finite difference approximations of consistent order truncation error reduced the overall error of the numerical solution in a typical problem by a factor of five for truncation error Oh^2. For the present numerical analysis, a computational mesh of N = 50 was used for all cases. Now with the interval h = 1/N = 0.02, examination of the consistent fourth order truncation error term will yield an error of the order of $5x10^{-9}$ for all four derivatives approximated, since the magnitude of the normalized functions and derivatives are of order one. For an interval size of h = 0.02, a consistent truncation error Oh^4 should significantly reduce the overall error of the solution.

Special problems arise in the derivation of the finite difference equations at stations near the boundaries. Several approaches can be taken to approximate the derivatives at these stations, which include backward, forward and central finite differencing techniques. Unfortunately, the central method previously discussed will involve imaginary stations outside the boundaries for stations close to the axis and wall, i.e., stations i = -1, -2, -3, and N+1, N+2, N+3 are involved in the fourth derivative approximation at station i = 1 and i = N, respectively. Unlike the boundary conditions for many problems well suited for this

type of numerical analysis, the vorticity transport equation boundary conditions for the pipe flow problem are very complicated by comparison. The boundary conditions for each angular wave number are different; some involve additional parameters while others contain the eigenvalue γ. In other words, the conditions at the imaginary stations for conventional problems can normally be expressed in terms of conditions at the real stations in the computational mesh. It is therefore impractical to use the central method for the pipe flow problem because of the complicated nature of the boundary conditions.

The method of forward differencing is used near the pipe axis and backward differencing is used near the wall. These two differencing methods will be referred to, collectively, as noncentral finite differencing techniques. It will be shown later that the two methods are essentially equivalent. A rigorous derivation of the noncentral finite difference equations is based on the Taylor series (power series) expansion of the function at a point. Since the governing equations contain the fourth order derivative of Q(r), the derivation for this case is examined in detail here.

The Taylor series approximation of $Q_1(r)$ at station i = 1 appears as equation (5-9).

$$Q_1 = Q(0) + DQ(0)\,\frac{h}{2} + D^2Q(0)\,\frac{(\frac{h}{2})^2}{2!} + D^3Q(0)\,\frac{(\frac{h}{2})^3}{3!} + D^4Q(0)\,\frac{(\frac{h}{2})^4}{4!}$$

$$+ D^5Q(0)\,\frac{(\frac{h}{2})^5}{5!} + D^6Q(0)\,\frac{(\frac{h}{2})^6}{6!} + D^7Q(0)\,\frac{(\frac{h}{2})^7}{7!} + \ldots \qquad (5\text{-}9)$$

The approximation for Q_i at stations i = 1, 2, 3, 4, 5, 6 are expressed in convenient matrix format below,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{1}{2})^5}{5!} & \frac{(\frac{1}{2})^6}{6!} & \frac{(\frac{1}{2})^7}{7!} \\ 1 & \frac{3}{2} & \frac{(\frac{3}{2})^2}{2!} & \frac{(\frac{3}{2})^3}{3!} & \frac{(\frac{3}{2})^4}{4!} & \frac{(\frac{3}{2})^5}{5!} & \cdot & \cdot \\ 1 & \frac{5}{2} & \frac{(\frac{5}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{7}{2} & \frac{(\frac{7}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{9}{2} & \frac{(\frac{9}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{11}{2} & \frac{(\frac{11}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{6\times 8} \begin{Bmatrix} Q(0) \\ hDQ(0) \\ h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} + \{E_1\}\, h^8D^8Q(0) \qquad (5\text{-}10)$$

where the dots represent matrix elements that can be easily deduced.

Since the fourth order derivative of Q(r) is present in the vorticity transport equations, there will be two boundary conditions specified at the axis and two at the wall. It can be shown that two columns of the matrix in equation (5-10) can be eliminated since two of the functions in the column vector are either zero or can be expressed in terms of additional parameters. The matrix, which will be denoted as [AA], now becomes a square 6 X 6 matrix. Notice that the column vector $\{E_1\}$ represents the last nonvanishing term in the Taylor series. The approximation for the first derivative of $Q_i(r)$ at station i = 1 is given by,

$$DQ_1 = DQ(0) + D^2Q(0)\,\frac{h}{2} + D^3Q(0)\,\frac{(\frac{h}{2})^2}{2!} + D^4Q(0)\,\frac{(\frac{h}{2})^3}{3!} + D^5Q(0)\,\frac{(\frac{h}{2})^4}{4!}$$

$$+ D^6Q(0)\,\frac{(\frac{h}{2})^5}{5!} + D^7Q(0)\,\frac{(\frac{h}{2})^6}{6!} + \ldots\ldots \qquad (5\text{-}11)$$

The Taylor series expansion for the second, third, and fourth derivative can be found by successive deferentiation of equation (5-11). The first four derivatives of Q_i at station i = 1 can be expressed in matrix format as,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{1}{2})^5}{5!} & \frac{(\frac{1}{2})^6}{6!} \\ 0 & 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{1}{2})^5}{5!} \\ 0 & 0 & 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} \\ 0 & 0 & 0 & 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} \end{bmatrix}_{4\times 8}$$

$$\begin{Bmatrix} Q(0) \\ hDQ(0) \\ h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} + \{E_2\}\, h^8D^8Q(0) \qquad (5\text{-}12)$$

It can be shown here as well that two columns of the matrix in equation (5-12) can be eliminated so that the matrix denoted [CC] is now a 4 X 6 matrix. The column vector $\{E_2\}$ also represents the last non-vanishing term in the Taylor series. It can be seen that the approximations for the derivatives of Q_i at stations i = 2 and i = 3 can be determined by replacing (1/2) in the [CC] matrix with (3/2) and (5/2), respectively.

To show how the noncentral finite difference approximations near the axis are formulated, the case for angular wave number $n \geq 6$ is examined. The boundary conditions at the axis for the function Q are,

$$Q(0) = 0 \qquad \text{and} \qquad DQ(0) = 0 \tag{5-13}$$

Applying these conditions to equations (5-10) and (5-12) will eliminate the first two columns of the [AA] and [CC] matrices since the first two terms of the column vectors vanish. The truncation error terms $\{E_1\}$ and $\{E_2\}$ are very small quantities and are neglected at this point. In shorthand notation, equation (5-10) now becomes,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \underset{6\times6}{[AA]} \begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} \tag{5-14}$$

and equation (5-12) becomes,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \underset{4\times6}{[CC]} \begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} \tag{5-15}$$

Equation (5-14) is multiplied through $[AA]^{-1}$ and becomes,

$$\begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} = \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} \tag{5-16}$$

and is then substituted into equation (5-15) giving,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \underset{4\times6}{[CC]} \quad \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} \tag{5-17}$$

Now, the fourth derivative of Q_i at station i = 1 is approximated in terms of the discrete functions of Q at the interior mesh points with truncation error Oh^4. The values on the bottom row of the resulting 4 X 6 matrix after multiplication in equation (5-17) are the coefficients of the column vector Q_1 through Q_6 for D^4Q_1. In order to have consistent truncation error Oh^4, the last row and last column of the [AA] and [CC] matrices are deleted for the third derivative of Q. Matrix [AA] becomes a 5 X 5 and is now inverted and matrix [CC] becomes a 3 X 5. The values on the bottom row of the resulting 3 X 5 matrix are

the coefficients of Q_1 through Q_5 which approximates D^3Q_1. The rows and columns of matrices [AA] and [CC] are successively reduced to determine the coefficients for D^2Q_1 in terms of Q_1 through Q_4, and DQ_1 in terms of Q_1 through Q_3 so that consistent truncation error Oh^4 is preserved throughout. The noncentral finite difference equation for D^4Q_1 will now appear as,

$$D^4Q_1 = \frac{1}{h^4}\,(A_1Q_1 + A_2Q_2 + A_3Q_3 + A_4Q_4 + A_5Q_5 + A_6Q_6) + Oh^4 \qquad (5\text{-}18)$$

where A_1 through A_6 are the coefficients taken from the bottom row of the resulting matrix in equation (5-17)

In a similar manner, by imposing the required boundary conditions, this process can be carried out to determine the noncentral finite difference equations for the first two derivatives of P(r) at several points near the boundaries as required.

It is important to note here that the noncentral finite difference approximations near the wall can be formulated by using the same techniques as those used near the axis. Once again taking the case for angular wave number $n \geqq 6$, the boundary conditions at the pipe wall for the function Q are,

$$Q(1) = 0 \qquad \text{and} \qquad DQ(1) = 0 \qquad (5\text{-}19)$$

Rather than set up complete new [AA] and [CC] matrices at different values of r in the Taylor series expansion near the wall, assume that the boundary conditions in equation (5-19) are imposed at the axis,

r = 0. Notice that the boundary conditions in equation (5-19) are now identical to the boundary conditions in equation (5-13). The method for computing the coefficients of the noncentral finite difference approximations at the wall can be carried out as discussed previously and, in this case, have already been computed. It can be shown that the A_i coefficients of equation (5-18) are applicable to D^4Q_N at the wall but will appear in reverse order. The noncentral finite difference equation for D^4Q_N is given by,

$$D^4Q_N = \frac{1}{h^4}\left(A_6Q_{N-5} + A_5Q_{N-4} + A_4Q_{N-3} + A_3Q_{N-2} + A_2Q_{N-1} + A_1Q_N\right) + Oh^4 \tag{5-20}$$

The method for determining the coefficients for the remaining derivatives of Q and P near the wall proceed as though the boundary conditions were imposed at the axis; the coefficients are computed, their order reversed and coefficients of the odd order derivatives undergo a sign change. Order reversal and sign changes are accomplished in part II of the main investigative program when the precomputed noncentral finite difference coefficients are read in as data.

As can be seen from the central finite difference approximations of equations (5-5) through (5-8), the third and fourth derivatives of a function require three points either side of the central point being approximated. The first and second derivatives require only two points either side of the central point. Therefore the central finite difference approximations are applicable only to interior stations i = 4 through N-3 for D^4Q_i and D^3Q_i, and stations i = 3 through N-2 for D^2Q_i,

DQ_i, D^2P_i, and DP_i. This leaves either two or three points immediately adjacent to the boundaries to be approximated by the noncentral finite difference equations just developed.

Since the central finite difference approximations are applicable to the above mentioned interior mesh points and are valid for all the derivatives of P(r) and Q(r) as well, the coefficients are computed only once in part I of the main investigative program in double precision format. The coefficients of the noncentral finite difference approximations, which are dependent on the boundary conditions for each angular wave number, are computed in an independent computer program . The resulting noncentral coefficients for the derivatives of P(r) and Q(r) at mesh points near the boundaries are also computed in double precision format and are read into part II of the main investigative program as data.

C. SPECIFIC METHODS FOR n = 0, 1, AND 6

The vorticity transport equations have now been expanded into a set of discrete simultaneous equations with respect to a half station computational mesh of N points (stations). The derivatives of the governing equations were approximated by central and noncentral finite difference coefficients, with consistent fourth order truncation error, in terms of the discrete unknown functions of the system. The convenient matrix format of the eigensystem in equation (5-3) can now be solved on a digital computer using a specialized Fortran subroutine, which will be discussed later. It is important to understand the composition of the [A] and [B] matrices and the column vector of unknowns

{X}. The problem setup is examined in detail for each angular wave number investigated. In the following discussions, the vorticity transport equations are referred to in the transformed state after the change of variables to P(r) and Q(r) have been made. The boundary conditions in equations (3-17), (3-18), and (3-23), and the coefficients in Appendix B also apply.

For the case n = 0, the pipe flow problem becomes greatly simplified in that the equations become uncoupled and only the solution for Q(r) is investigated for reasons explained earlier. The governing equation in terms of Q(r) is represented by equation (2-38). At the outset, it was determined that a mesh size of N = 50 was sufficient to produce a satisfactory numerical solution of the problem. It was necessary to determine when the solution was relatively independent of mesh size. Preliminary trial computations were performed for various mesh sizes, N = 25 to 100. From N = 45 to 100, the solution of the equation for n = 0 did not vary appreciably. Additionally, the solution was relatively independent of Reynolds numbers below 15,000 for mesh sizes of about 50. In order to reduce computation time and the eigensystem matrix size, a mesh of N = 50 was used in this analysis.

In general for angular wave number n = 0, the [A] and [B] matrices are 50 X 50 and appear as banded diagonal matrices. For a consistent order truncation error Oh^4 used here, the derivative of highest order requires the greatest band width. From the governing equation in Q, it can be seen that the highest order derivative in [A] is D^4Q and in [B] is D^2Q. At stations near the boundaries, the noncentral finite difference coefficients require a band width of six in matrix [A] and four in

matrix [B]. At interior stations, the central finite difference coefficients require a band width of seven for [A] and five for [B]. Therefore the [A] and [B] matrices of equation (5-3) contain nonzero elements as shown in Figures 5-3 and 5-4. The X elements contain non-central finite difference coefficients and the 0 elements contain central difference coefficients. The X and θ symbols denote the station at which the approximation is made. The composition of the column vector {X} can also be seen in Figures 5-3 and 5-4.

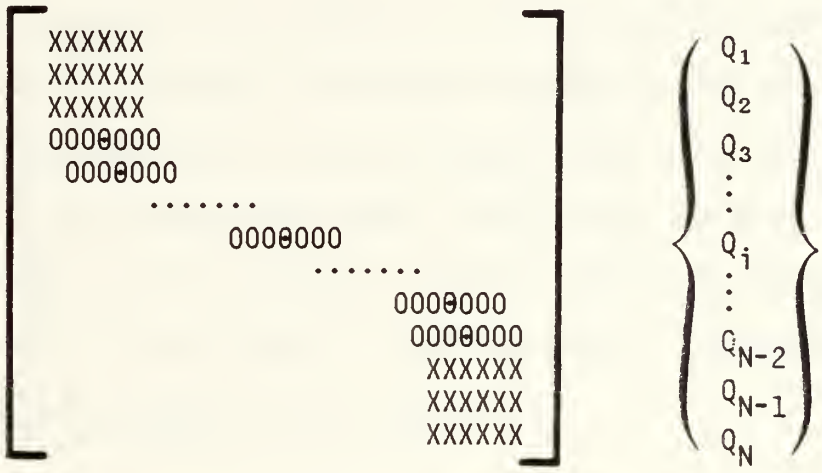


Figure 5-3. [A] Matrix for n = 0

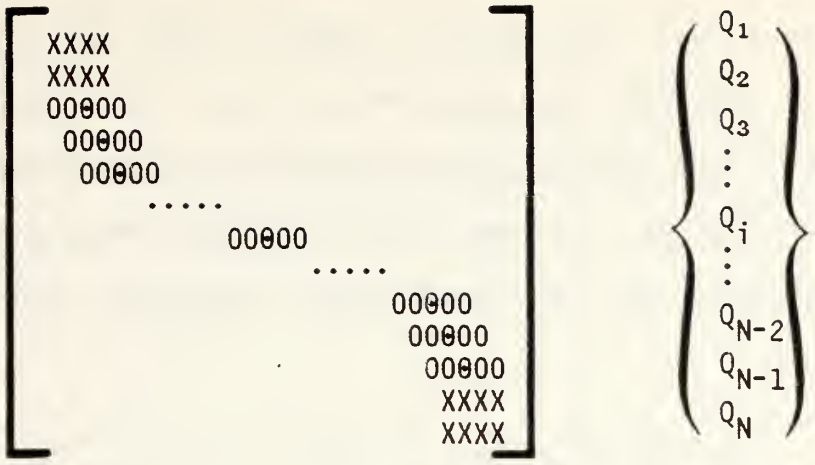


Figure 5-4. [B] Matrix for n = 0

The actual values of the elements of the [A] and [B] matrices are computed in part IV of the main investigative program. The calculations are rather lengthy and tedious and are accomplished in iterative loops in part IV. To save computation time, calculations are made only for the nonzero elements of the matrices. Recall that the governing equation in Q(r) for n = 0 was given in equation (2-43). In discrete form for the ith station, equation (2-43) appears as,

$$M'_{4_i} D^4Q_i + M'_{3_i} D^3Q_i + M'_{2_i} D^2Q_i + M'_{1_i} DQ_i + M'_{0_i} Q_i =$$

$$\gamma(N'_{2_i} D^2Q_i + N'_{1_i} DQ_i + N'_{0_i} Q_i) \qquad (5\text{-}21)$$

where the primed quantities are element (2,2) of the corresponding transformed vorticity transport equation matrices given in Appendix B, Part A. At station i = 1, the top row of the [A] matrix will have six elements and the top row of [B] will have four elements.

For example, the matrix elements for the first row of [A] can be found as follows, where the a_i coefficients are analogous to the coefficients A_i of equation (5-18) and are equal to $a_i = A_i/h^4$. Similarly $b_i = B_i/h^3$, $c_i = C_i/h^2$, and $d_i = D_i/h$, where b_i, c_i, and d_i are the noncentral finite difference coefficients of the remaining three derivatives.

$$A_{1,1}\{Q_1\} = (M'_{4_1} a_1 + M'_{3_1} b_1 + M'_{2_1} c_1 + M'_{1_1} d_1 + M'_{0_1})\{Q_1\} \quad (5\text{-}22)$$

$$A_{1,2}\{Q_2\} = (M'_{4_1} a_2 + M'_{3_1} b_2 + M'_{2_1} c_2 + M'_{1_1} d_2)\{Q_2\} \quad (5\text{-}23)$$

$$A_{1,3}\{Q_3\} = (M'_{4_1} a_3 + M'_{3_1} b_3 + M'_{2_1} c_3 + M'_{1_1} d_3)\{Q_3\} \quad (5\text{-}24)$$

$$A_{1,4}\{Q_4\} = (M'_{4_1} a_4 + M'_{3_1} b_4 + M'_{2_1} c_4)\{Q_4\} \quad (5\text{-}25)$$

$$A_{1,5}\{Q_5\} = (M'_{4_1} a_5 + M'_{3_1} b_5)\{Q_5\} \quad (5\text{-}26)$$

$$A_{1,6}\{Q_6\} = (M'_{4_1} a_6)\{Q_6\} \quad (5\text{-}27)$$

where $A_{1,1}$ is the matrix element in the first row and the first column of [A], and so forth. This procedure is carried out for N equations along the computational mesh until all the nonzero matrix elements of [A] and [B] are computed.

Derivation of the noncentral finite difference coefficients for other angular wave numbers is fairly straightforward when accomplished by the methods outlined in the previous section. The boundary conditions for Q at the axis for n = 0 are,

$$DQ(0) = 0 \qquad \text{and} \qquad D^3Q(0) = 0 \tag{5-28}$$

After imposing these conditions, the second and fourth columns of the matrices in equations (5-10) and (5-12) can be eliminated since the corresponding terms in the column vectors vanish. The coefficients are then computed as described before in an independent computer program and then read into the main investigative program as data. Notice that the bondary conditions for the function Q at the wall are the same for $n = 0$ and $n \geqq 6$. Therefore, the noncentral finite difference coefficients need to be computed only once.

The case for angular wave number $n \geqq 6$ will be examined next since it more nearly represents the general case. Since the vorticity transport equations remain coupled for $n \geqq 6$ and no additional parameters are introduced, a system of 2N equations in 2N unknowns will result. For a mesh size of 50, the [A] and [B] matrices become 100 X 100 and are solved in the form of equation (5-3). The first N equations of the system are represented by the discrete form of equation (2-37), which after transformation by the appropriate change of variables corresponds to the top equation of (5-1). The second N set of equations are represented by the discrete form of equation (2-38), which after transformation by the appropriate change of variables corresponds to the bottom equation of (5-1). As a matter of convenience for programming, the [A] and [B] matrices are partitioned into four 50 X 50 quadrants, which are individually banded diagonal matrices. The properties of the [A] and [B] matrices for $n \geqq 6$ are summarized in Table 5-1.

Table 5-1. Properties of the [A] and [B] Matrices for $n \geq 6$

MATRIX	QUADRANT	HIGHEST ORDER DERIVATIVE PRESENT	NONCENTRAL COEFFICIENTS REQUIRED	*	CENTRAL COEFFICIENTS REQUIRED
A	I	D^2P	5	2	5
A	II	D^3Q	5	3	7
A	III	D^2P	5	2	5
A	IV	D^4Q	6	3	7
B	I	P	-	-	1
B	II	DQ	3	2	5
B	III	P	-	-	1
B	IV	D^2Q	4	2	5

* Number of stations near the boundaries where the noncentral finite difference approximations apply.

It can be deduced from the data given in Table 5-1 that the [A] and [B] matrices will contain nonzero elements as shown in Figures 5-5 and 5-6. The column vector {X} of unknowns is also given.

The method for determining the noncentral finite difference coefficients for $n \geq 6$ was discussed in a previous example.

The case for $n = 1$ is unique in that an additional parameter H(0) is introduced by the change of variables and a pair of coupled boundary conditions at the axis in equation (3-18b) contain the eigenvalue γ. These peculiarities are significant factors in correctly setting up the problem. Recall that in the coupled governing equations (2-47) the additional parameter H(0) appears explicitly and two additional column

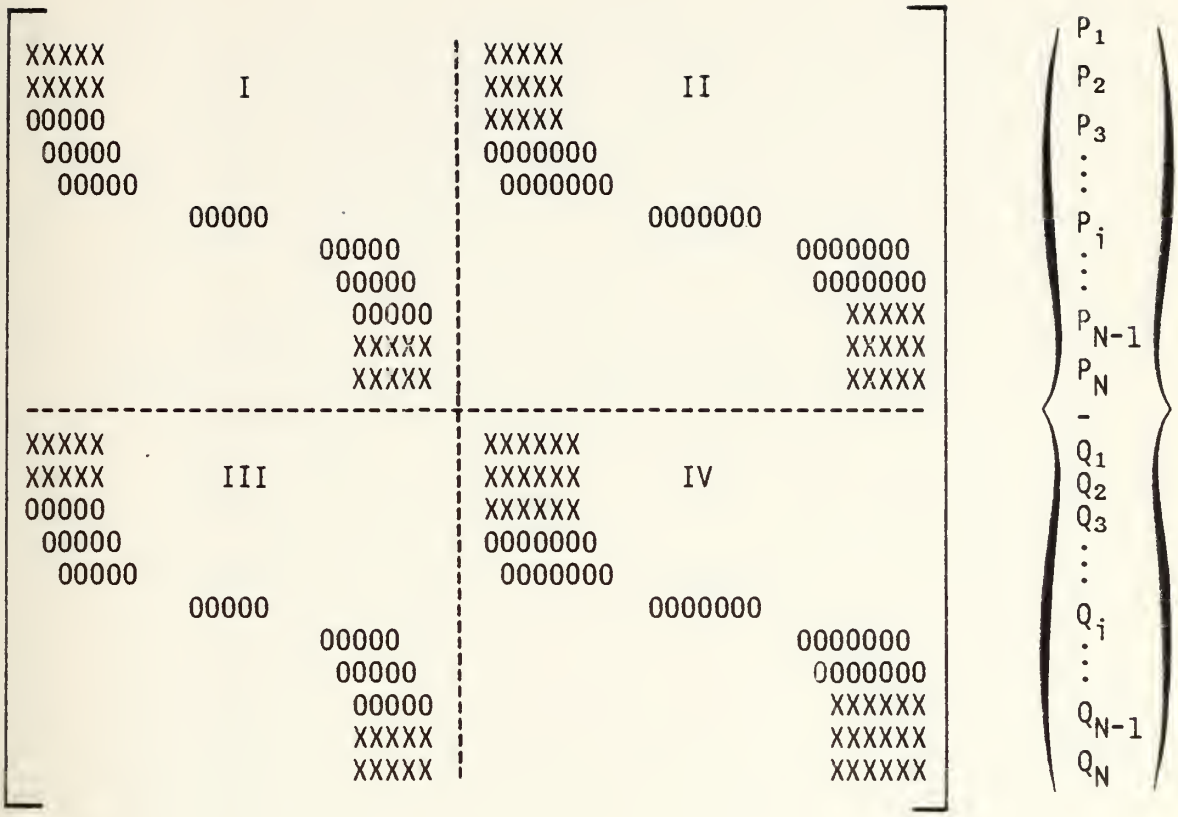


Figure 5-5. [A] Matrix for $n \geq 6$

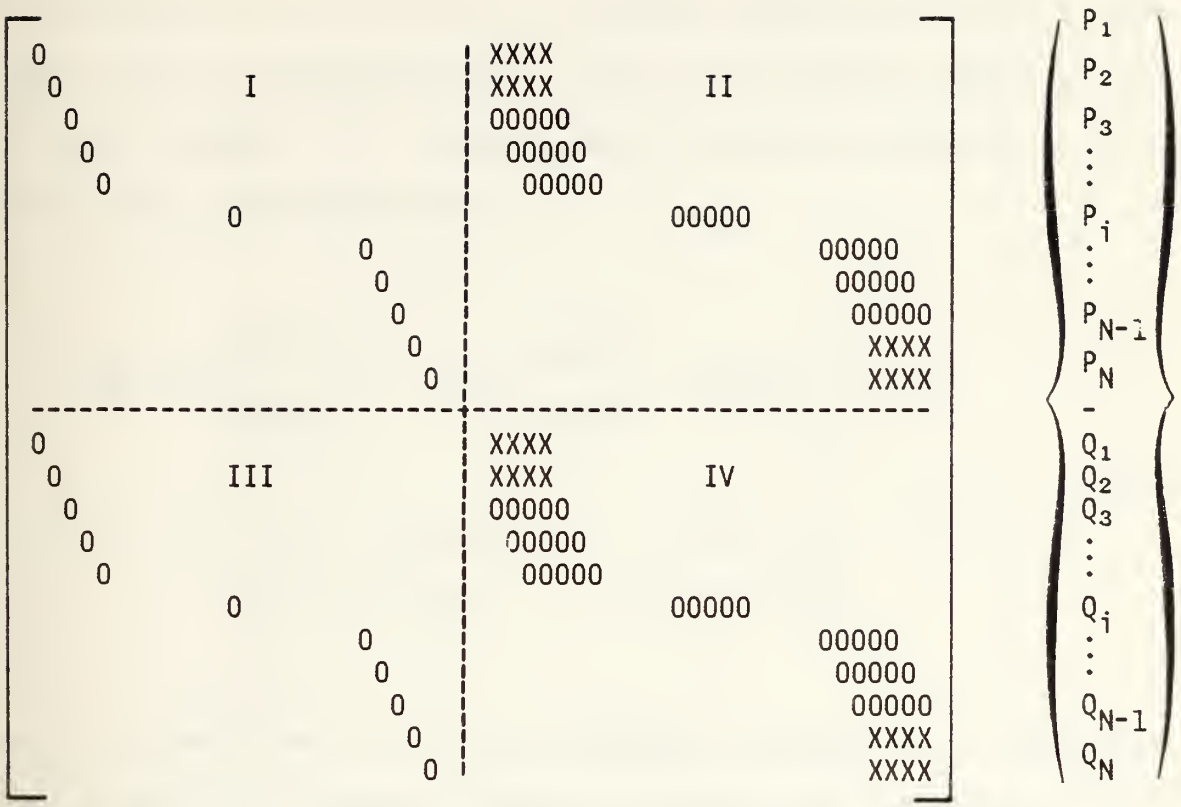


Figure 5-6. [B] Matrix for $n \geq 6$

matrices are required in the system of equations. This was due to the change of variables introduced by equations (2-44). The appearance of the eigenvalue γ in the pair of boundary equations at the axis comes about from the rigorous method in which the boundary conditions were derived in Chapter III. The coupled axis boundary conditions in equation (3-18b) are repeated here,

$$-\frac{96}{Re}[C_7]\begin{Bmatrix} D^2P(0) \\ D^2Q(0) \end{Bmatrix} + 2i\alpha[C_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + i\alpha[C_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)$$

$$= \gamma\left(2\,[D_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + \alpha^2\,[D_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)\right) \tag{5-29}$$

where the coefficient matrices are contained in Appendix C. Notice that the parameter H(0) appears explicitly in these two equations as well. Additionally, the parameters $D^2P(0)$, $D^2Q(0)$, $P(0)$, and $Q(0)$ are also introduced. Obviously, this poses some problems since there are only two additional equations and five unknown parameters. H(0) can be used since it appears in the 2N discrete equations of the system. Q(0) is chosen as the second variable as a matter of convenience. The remaining three functions can be approximated in terms of discrete functions of P_i in the case for $D^2P(0)$ and $P(0)$, and Q_i and $Q(0)$ in the case of $D^2Q(0)$. By referring to the matrix form of the Taylor series expansion of a function at a point in equation (5-10), it can be seen how the approximations of these functions are made. For the present situation where

DP(0) = 0, eliminate the second column of the matrix and equation (5-10) will become,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix} = \underset{5\times5}{[AA]} \begin{Bmatrix} P(0) \\ h^2D^2P(0) \\ h^3D^3P(0) \\ h^4D^4P(0) \\ h^5D^5P(0) \end{Bmatrix} + Oh^6 \qquad (5\text{-}30)$$

Recall that the matrix in equation (5-30) is denoted [AA] as before. After multiplying each side of the equation by $[AA]^{-1}$, the result becomes,

$$\begin{Bmatrix} P(0) \\ h^2D^2P(0) \\ h^3D^3P(0) \\ h^4D^4P(0) \\ h^5D^5P(0) \end{Bmatrix} = \underset{5\times5}{[AA]^{-1}} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix} - Oh^6 \qquad (5\text{-}31)$$

Now P(0) is approximated by the coefficients in the first row of $[AA]^{-1}$ and $D^2P(0)$ is approximated by the coefficients in the second row of $[AA]^{-1}$ divided by h^2, both expressed in terms of P_1 through P_5. Notice that the approximation for $D^2P(0)$ has truncation error Oh^4 and P(0) has truncation error Oh^6, which is acceptable in this analysis.

The approximation for $D^2Q(0)$ is similar except that Q(0) will appear explicitly in the approximations with DQ(0) = 0. Equation (5-10) now becomes,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \{Q(0)\} + \underset{6\times6}{[AA]} \begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} + Oh^8 \qquad (5\text{-}32)$$

After rearranging, equation (5-32) is multiplied through by $[AA]^{-1}$ and becomes,

$$\begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} = \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 - Q(0) \\ Q_2 - Q(0) \\ Q_3 - Q(0) \\ Q_4 - Q(0) \\ Q_5 - Q(0) \\ Q_6 - Q(0) \end{Bmatrix} - Oh^8 \qquad (5\text{-}33)$$

The coefficients for $D^2Q(0)$ are taken from the first row of $[AA]^{-1}$ and divided by h^2. $D^2Q(0)$ is now expressed in terms of $Q(0)$ and Q_1 through Q_6 and will have a truncation error Oh^6.

The two additional boundary condition equations involving γ can now be solved in the eigensystem of equations with $H(0)$ and $Q(0)$ appearing explicitly as the two required unknowns. It is convenient to express the $[C_9]$ and $[D_9]$ matrices of equation (5-29) in slightly different form,

$$[C_9] \begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0) \quad \text{as} \quad [C_9^*] \{H(0)\}$$

and (5-34)

$$[D_9] \begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0) \quad \text{as} \quad [D_9^*] \{H(0)\}$$

where $[C_9^*]$ and $[D_9^*]$ become 2 X 1 column matrices,

$$[C_9^*] = \begin{bmatrix} \left(\frac{\alpha^3}{Re} - 4i + 2i\alpha^2\right) \\ \left(-2\alpha^2 + \frac{i\alpha^3}{Re}\right) \end{bmatrix} \quad \text{and} \quad [D_9^*] = \begin{bmatrix} -i \\ 1 \end{bmatrix} \tag{5-35}$$

Substituting equations (5-35) into equation (5-29) yields,

$$-\frac{96}{Re} [C_7] \begin{Bmatrix} D^2P(0) \\ D^2Q(0) \end{Bmatrix} + 2i\alpha[C_8] \begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + i\alpha[C_9^*] \{H(0)\}$$

(5-36)

$$= \gamma \left(2 [D_8] \begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + \alpha^2 [D_9^*] \{H(0)\}\right)$$

The solution of the eigensystem of 2N + 2 equations in 2N + 2 unknowns is now accomplished in the form of equation (5-3). The appearance of the [A] and [B] matrices are identical to those for $n \geq 6$ in Figures 5-5 and 5-6 except for the addition of two columns to accommodate the parameters H(0) and Q(0), and two rows to accommodate the pair of boundary equations in (5-36). Nonzero elements are present in the [A] and [B] matrices for n = 1 as shown in Figures 5-7 and 5-8.

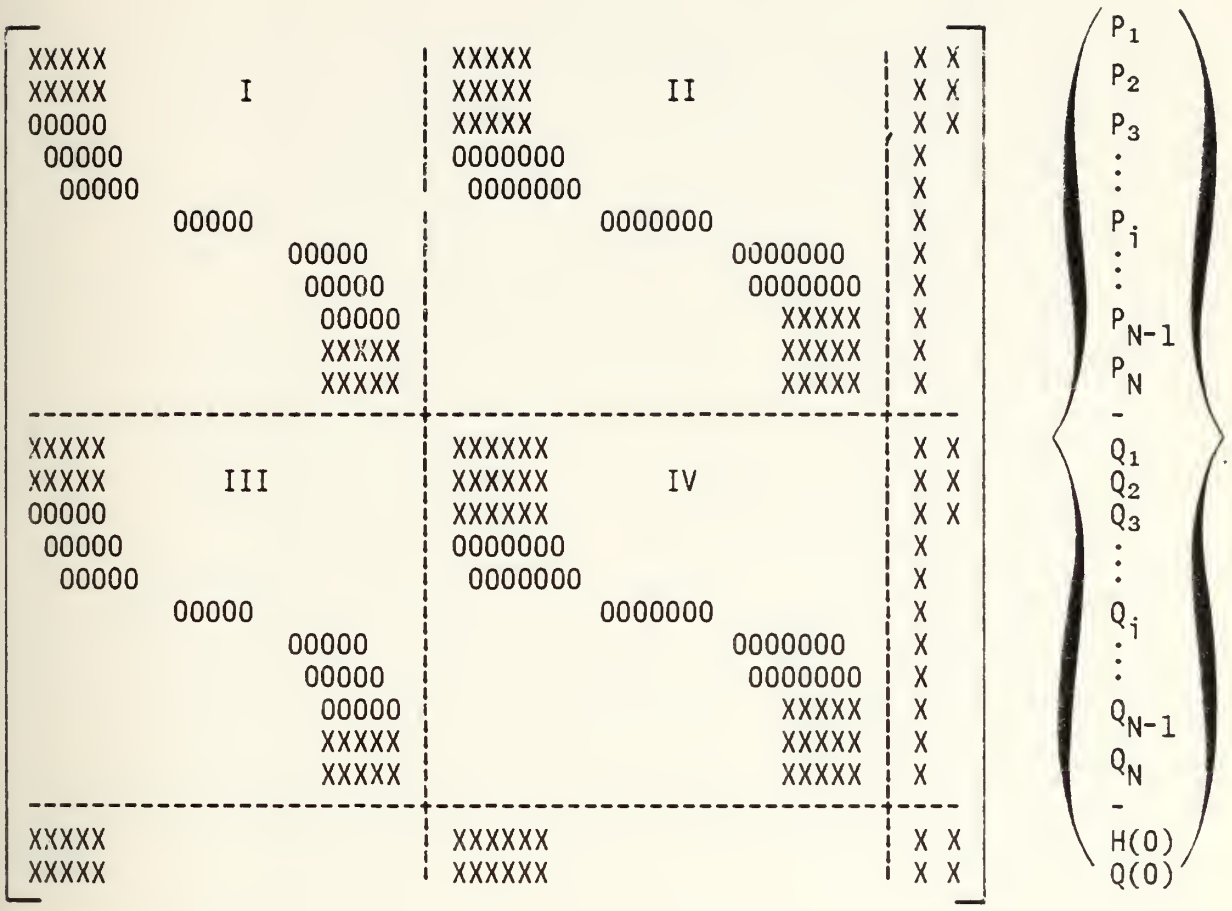


Figure 5-7. [A] Matrix for n = 1

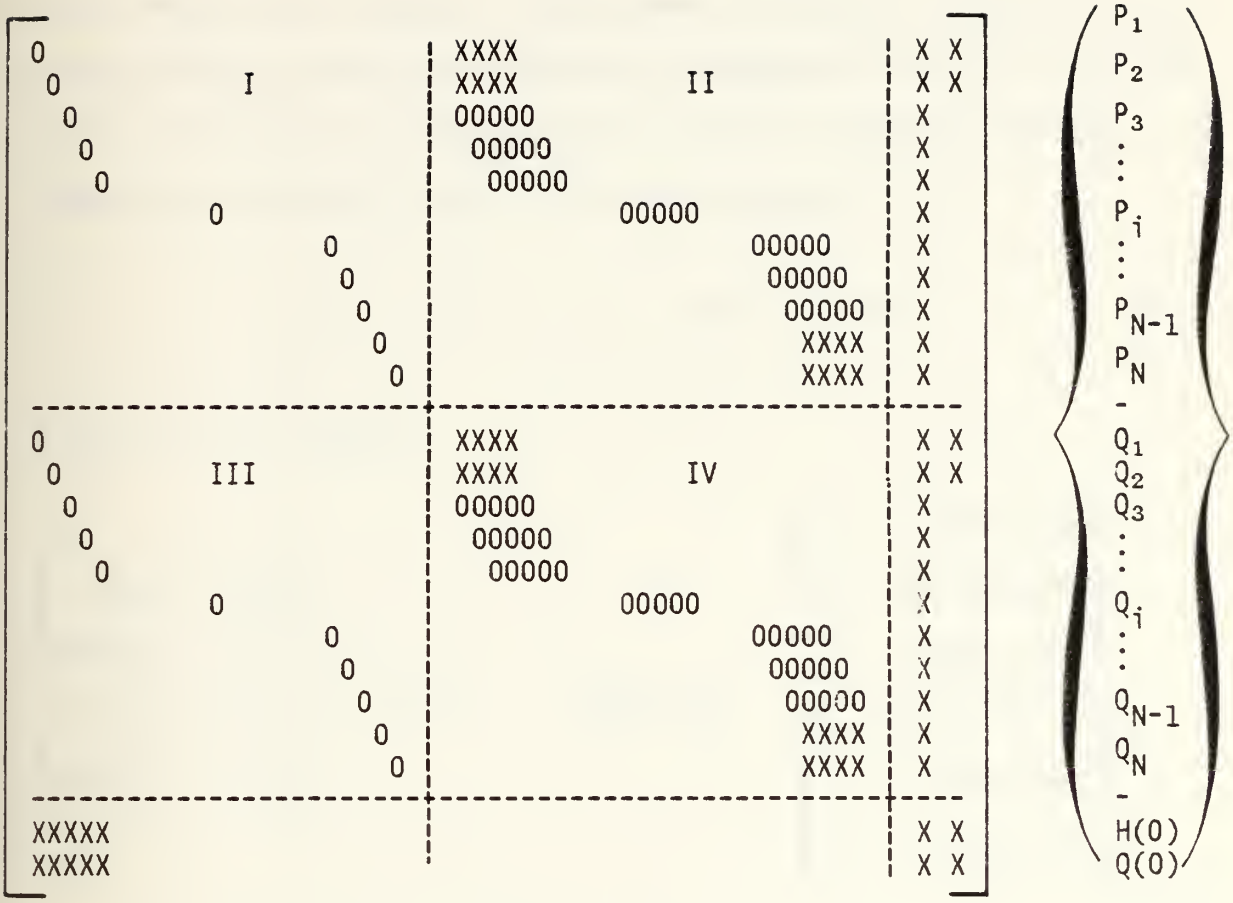


Figure 5-8. [B] Matrix for n = 1

Derivation of the noncentral finite difference coefficients for the derivatives of P and Q at the wall are straightforward as shown for the other angular wave numbers. The columns of the [AA] and [CC] matrices are eliminated based on the fact that parameters P(1), Q(1), and DQ(1) are all expressed in terms of H(0), as seen in equations (3-18c). By imposing the boundary conditions for Q at the wall,

$$Q(1) = -H(0) \quad \text{and} \quad DQ(1) = 2H(0) \tag{5-37}$$

equation (5-17) becomes,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \begin{bmatrix} 2h \\ 0 \\ 0 \\ 0 \end{bmatrix} \{H(0)\} + \underset{4\times6}{[CC]} \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 + (1-h)\,H(0) \\ Q_2 + (1-3h)\,H(0) \\ Q_3 + (1-5h)\,H(0) \\ Q_4 + (1-7h)\,H(0) \\ Q_5 + (1-9h)\,H(0) \\ Q_6 + (1-11h)\,H(0) \end{Bmatrix} \tag{5-38}$$

The coefficients in the resultant 4 X 6 matrix of equation (5-38) are in fact the same coefficients computed for $n \geq 6$ at the wall, and recall that they appear in reverse order when used at the wall. The details of the complete derivation of the noncentral finite difference coefficients near the boundaries is lengthy and not included here.

D. COMPUTER PROGRAM USEAGE

The solutions for the discrete transformed vorticity transport equations for the three angular wave numbers investigated are accomplished in the main investigative computer programs. Three separate programs were written rather than developing one program for the general case. The rationale for this is as follows: the changes of variables for each angular wave number $n \leqq 6$ are different, the boundary conditions are different, the equations uncouple for $n = 0$, additional parameters are introduced for some, and $n = 1$ has a coupled boundary equation that contains γ that must be dealt with separately in the setup of the problem. It would be practical to write a general program for angular wave numbers $n \geqq 6$, however.

For each specific angular wave number analyzed, the main computer program is divided into eight parts. Their functions are summarized below:

Part I — The central finite difference coefficient for the derivatives of P and Q are computed.

Part II — The noncentral finite difference coefficients for the derivatives of P and Q at the axis and wall are read in as data. Problem variables, Re and α, are read in as data.

Part III — Coefficients of the vorticity transport matrix equations are computed along the radial mesh.

Part IV — The [A] and [B] matrix elements are computed.

Part V — The eigenvalues and respective eigenvectors are computed in the subroutine EIGZC and the least stable eigenvalue is determined.

Part VI — Verification that the least stable eigenvalue and corresponding eigenvector satisfy the governing equations.

Part VII The normalized axial perturbation velocity is computed from the least stable eigenvalue and its eigenvector.

Part VIII The normalized axial perturbation velocity is plotted versus normalized pipe radius utilizing the Versatec plotter.

The numerical analysis of the pipe flow stability problem is centered around the subroutine EIGZC once the problem is properly set up. The eigensystem of equation (5-3) is solved by this subroutine which returns a set of N, 2N + 2, or 2N eigenvalues for n = 0, 1, or 6, respectively, and a corresponding set of normalized eigenvectors. Each eigenvalue and its eigenvector represent a solution of the governing equations. The primary objective of the main program and of this investigation is to determine the least stable eigenvalue which indicates flow characteristics. Recall that if γ_R is positive, the flow is unstable. For each value of α and Re chosen, the maximum algebraic value of γ_R is found, denoted as γ^*, to determine if instabilities exist in the flow field. The corresponding eigenvector of γ^*, denoted as $\{X^*\}$, is used to compute the incremental axial perturbation velocity along the radius. Plots of the normalized perturbation velocity versus normalized pipe radius have no significance in determining stability or instability of the flow but are indicators of the general nature and behavior of the flow field perturbations. Another independent program was written to plot Reynolds number stability contours using data taken from program results for n = 1 and 6. The subroutines used to make plots on Versatec plotter are PLOTG, standard Versatec plotting functions and TITLE1 to print titles and legends on the plots.

The numerical accuracy of the solution was determined by substituting γ^* and its eigenvector into the governing equations. For this purpose, the residual error vector $\{\varepsilon\}$ was computed by equation (5-39) and the magnitude of the error was examined.

$$\{\varepsilon\} = [A]\{X^*\} - \gamma^* [B]\{X^*\} \tag{5-39}$$

The Fortran subroutine EIGZC is contained in the International Mathematical and Statistical Library (IMSL) available in most large computer system libraries utilizing Fortran compilers. Specifically, the numerical analysis of the vorticity transport equations in the form of equation (5-3) was computed on the IBM 370/3033 (assigned processor) system with the Fortran H Extended compiler at the U.S. Naval Postgraduate School. All calculations were performed in full double precision format to improve the accuracy of the solution by minimizing truncation and round off errors inherently introduced by digital computations. The main investigative programs and other auxiliary programs were all written in Fortran programming language to be compatible with all versions of Fortran compilers. The computer program listings with applicable data sets are included after the appendices in this paper.

VI. RESULTS

A. RESULTS FOR n = 0

Results of the pipe flow stability problem are obtained from the solution of the linearized vorticity transport equations by the methods previously described. A generalized solution of the governing differential equations for fully developed, three-dimensional pipe flow has been accomplished for three angular wave numbers, n = 0, 1, and 6. The characteristics of the flow field perturbations for each angular wave number investigated are established by the parameters Re and α. The purpose of this numerical investigation was to determine flow stability at various Reynolds numbers and axial wave numbers for each angular wave number. Recall that the perturbation velocity vector $\vec{v}$ is defined in equation (2-27) as the curl of the velocity vector potential $\vec{W}$. This function is expressed in terms of G(r) and H(r) and a complex exponential form e^{X} in equations (2-28) and (2-29). After appropriate changes of variables were introduced, solutions were determined from the eigensystem of equations which consists of a set of eigenvalues and their corresponding normalized eigenvectors for fixed values of Re and α. The resulting perturbation quantities now appear in terms of the eigenfunctions P(r) and Q(r) and are represented in discrete form in the eigenvector {X} as P_i and Q_i, where i = 1, 2, 3.....N. As can be seen from equation (2-30), the real part of the complex eigenvalue γ_R will determine the growth or decay rate in time of the perturbation. Positive

values of γ_R indicate that the flow is unstable. The eigenvalue whose real part has the largest algebraic value, denoted as γ^*, is the least stable and is reported in these results to show stability trends.

Gawain [Ref. 13] reported the numerical results for angular wave number n = 0 for the solution of the governing equation in his paper. Those numerical results which incorporated a purely imaginary axial wave number and the advancements in the derivation of the boundary conditions at the pipe axis were duplicated in this investigation. They are essentially the same results except for variations in the fourth decimal place of the value of γ^*. This is due to the improved accuracy of the solution as a result of the numerical methods used here. The primary emphasis of this investigation is centered on determining the onset of flow instabilities at a theoretical critical Reynolds number near 1150. However, solutions were obtained at other Reynolds numbers to show stability trends and are presented in tabular and plotted form.

Stability data for angular wave number n = 0 for various combinations of Re and α are given in Table 6-1. The least stable eigenvalue solution of the governing equation is based on a computational mesh of N = 50. Table entries which contain dashes indicate that the solution of the governing equations was not sufficiently accurate to be included as valid data. Two facts become very clear in examining Table 6-1. First of all, the flow is stable at all Re and α for n = 0. Secondly, there is an asymptotic trend towards neutral stability for increasing Reynolds numbers. This second result agrees with experimental observations in that increasing Reynolds number has a destabilizing effect on the flow. It can also be seen that increasing the axial wave number has

Table 6-1. Stability Data for Angular Wave Number n = 0

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=1.0	α=2.0	α=4.0	α=8.0	α=16.0	α=32.0
300	-0.0879	-0.1544	-0.2360	-0.3399	-0.5152	-0.8665	-1.7770	-4.7194
575	-0.0459	-0.1196	-0.1685	-0.2429	-0.3614	-0.5831	-1.1122	-2.7236
1150	-0.0229	-0.0836	-0.1188	-0.1703	-0.2498	-0.3891	-0.6937	-1.5548
2300	-0.0115	-0.0591	-0.0838	-0.1197	-0.1737	-0.2633	-0.4429	-0.9099
4600	-0.0057	-0.0418	-0.0592	-0.0842	-0.1212	-0.1796	-0.2871	-
9200	-0.0029	-0.0295	-0.0418	-0.0593	-0.0846	-0.1225	-	-
18400	-0.0014	-0.0209	-0.0295	-0.0416	-0.0586	-	-	-

Note: Values are based on 50 mesh points.

a stabilizing effect on the flow. Normalized axial perturbation velocity plots versus pipe radius were made at various Re and α to examine the behavior of the perturbation quantities corresponding to the least stable eigenvalue solution. Figures 6-1 through 6-9, on the following pages, are representative of the behavior of the axial perturbation velocity. The activity is concentrated near the pipe axis for $n = 0$ and dies out quickly as r approaches 1.0. Additionally, the perturbation velocity plots were made in order to determine if mesh fineness was sufficient to obtain an accurate solution of the equations. Trial calculations were made for uniform mesh sizes up to 100 where the numerical results did not vary appreciably from those reported here.

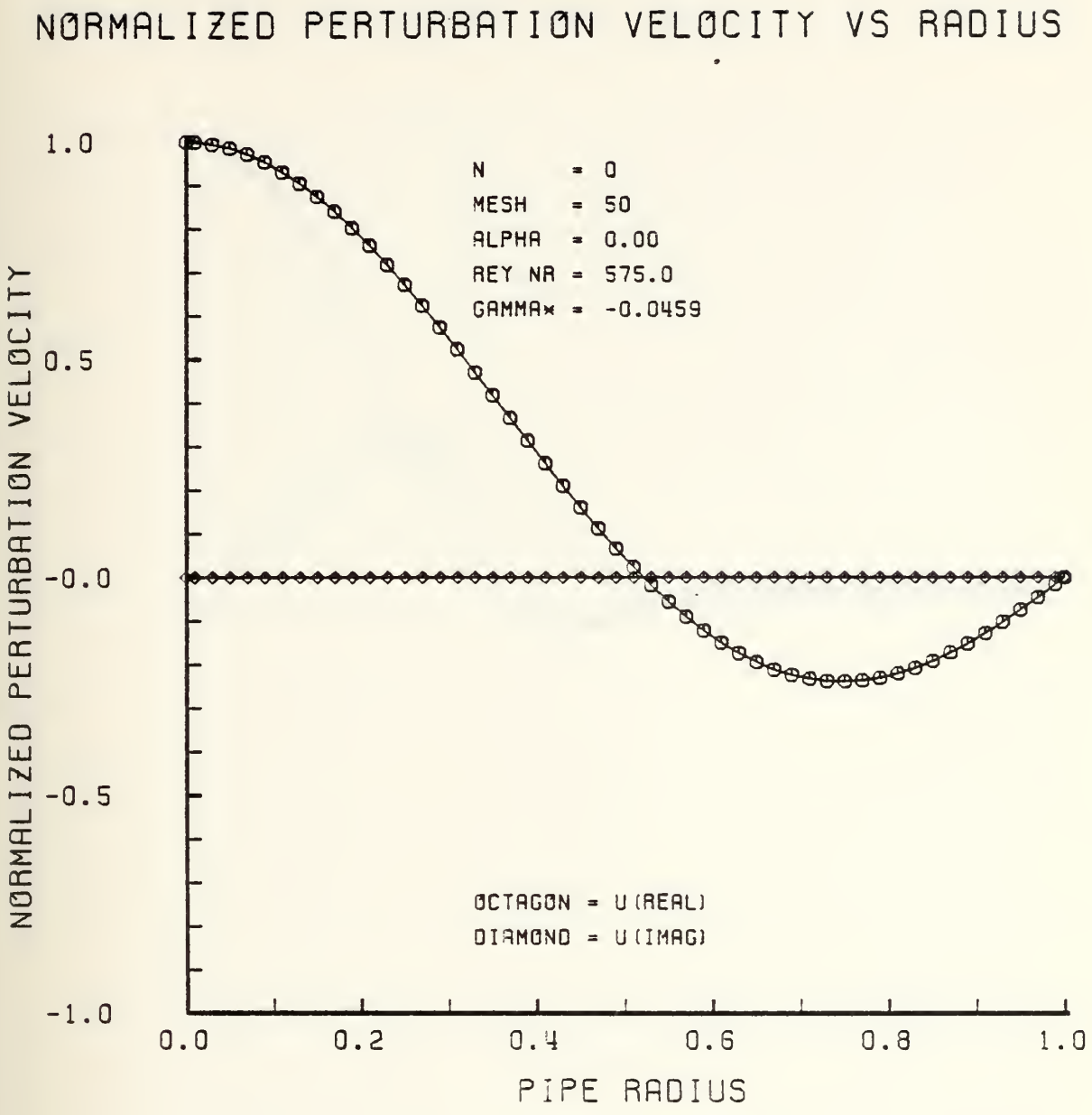

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 0.00
REY NR = 575.0
GAMMA× = -0.0459
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-1

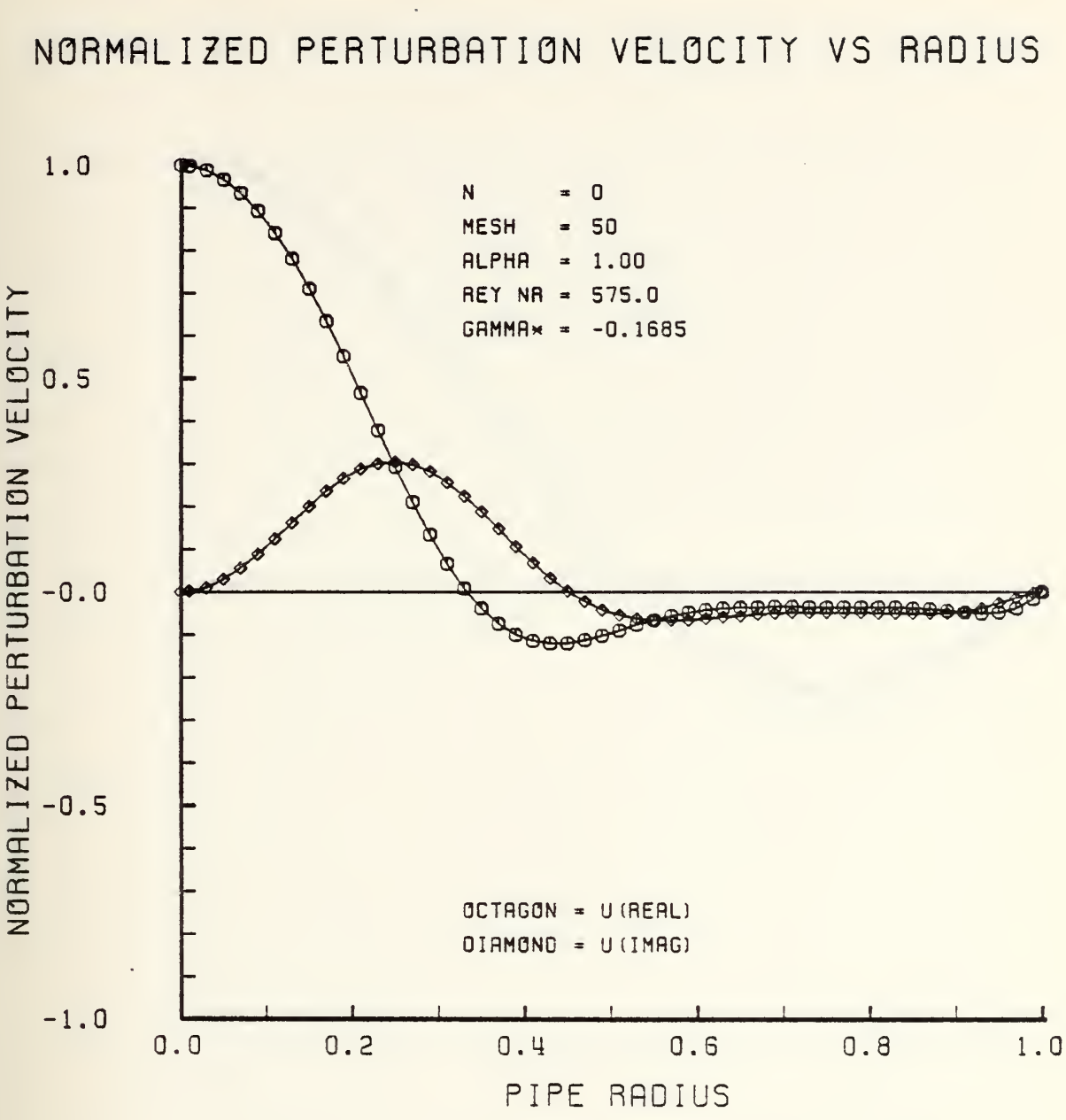
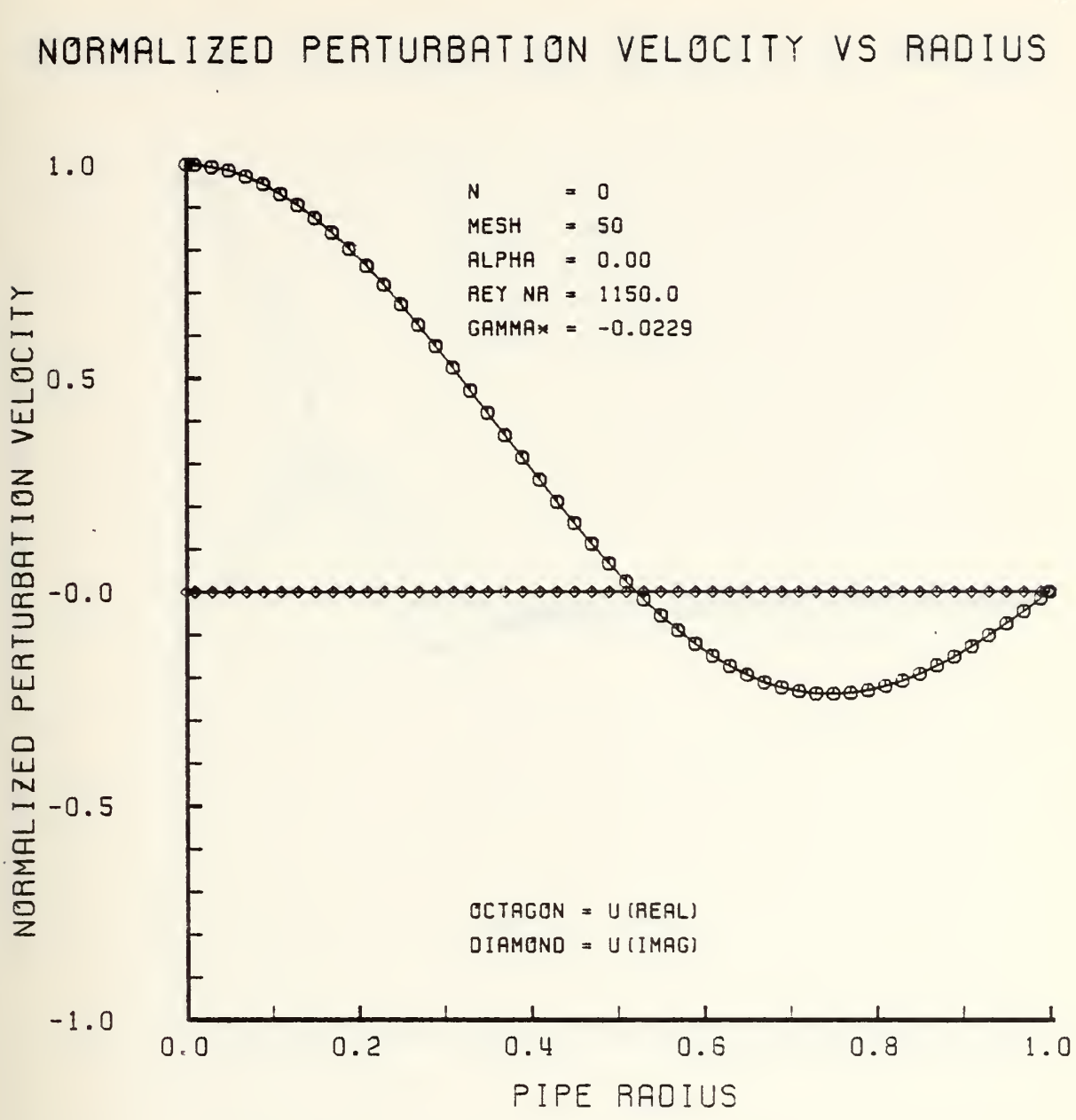

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 575.0
GAMMA× = -0.1685
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-2


NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 0.00
REY NR = 1150.0
GAMMA× = -0.0229
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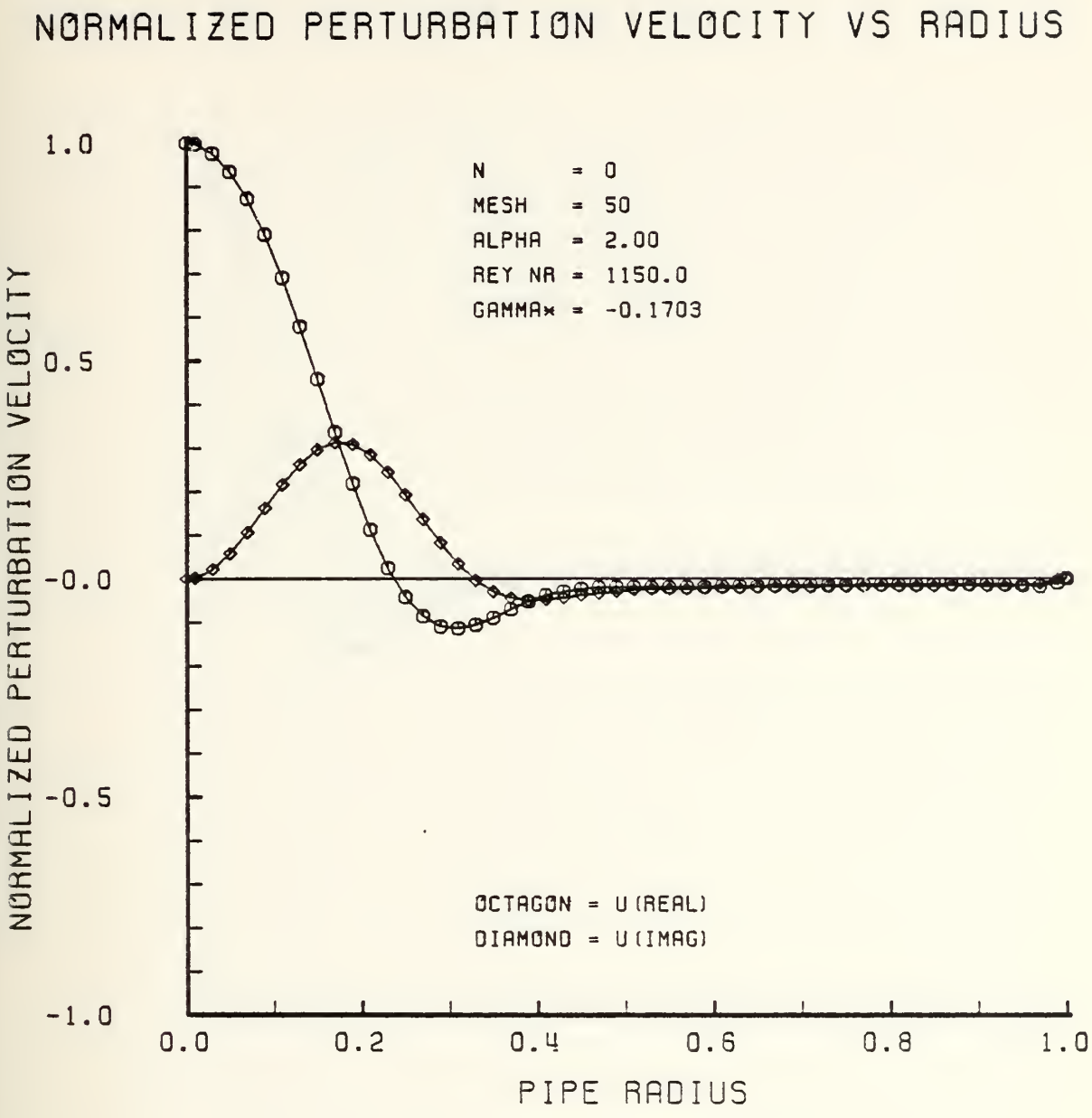
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-3


NORMALIZED PERTURBATION VELOCITY VS RADIUS
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MESH = 50
ALPHA = 2.00
REY NR = 1150.0
GAMMA* = -0.1703
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-4

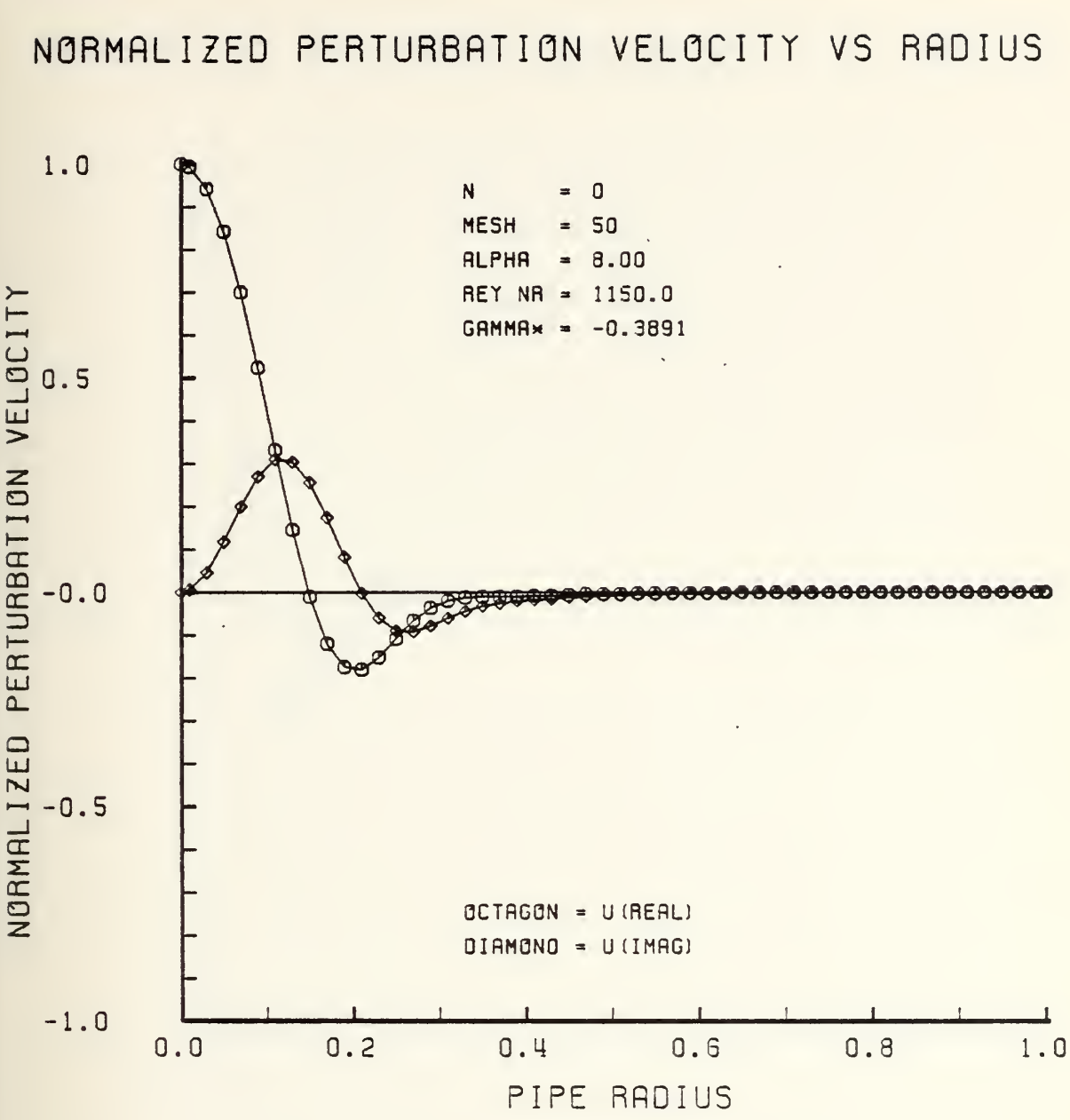
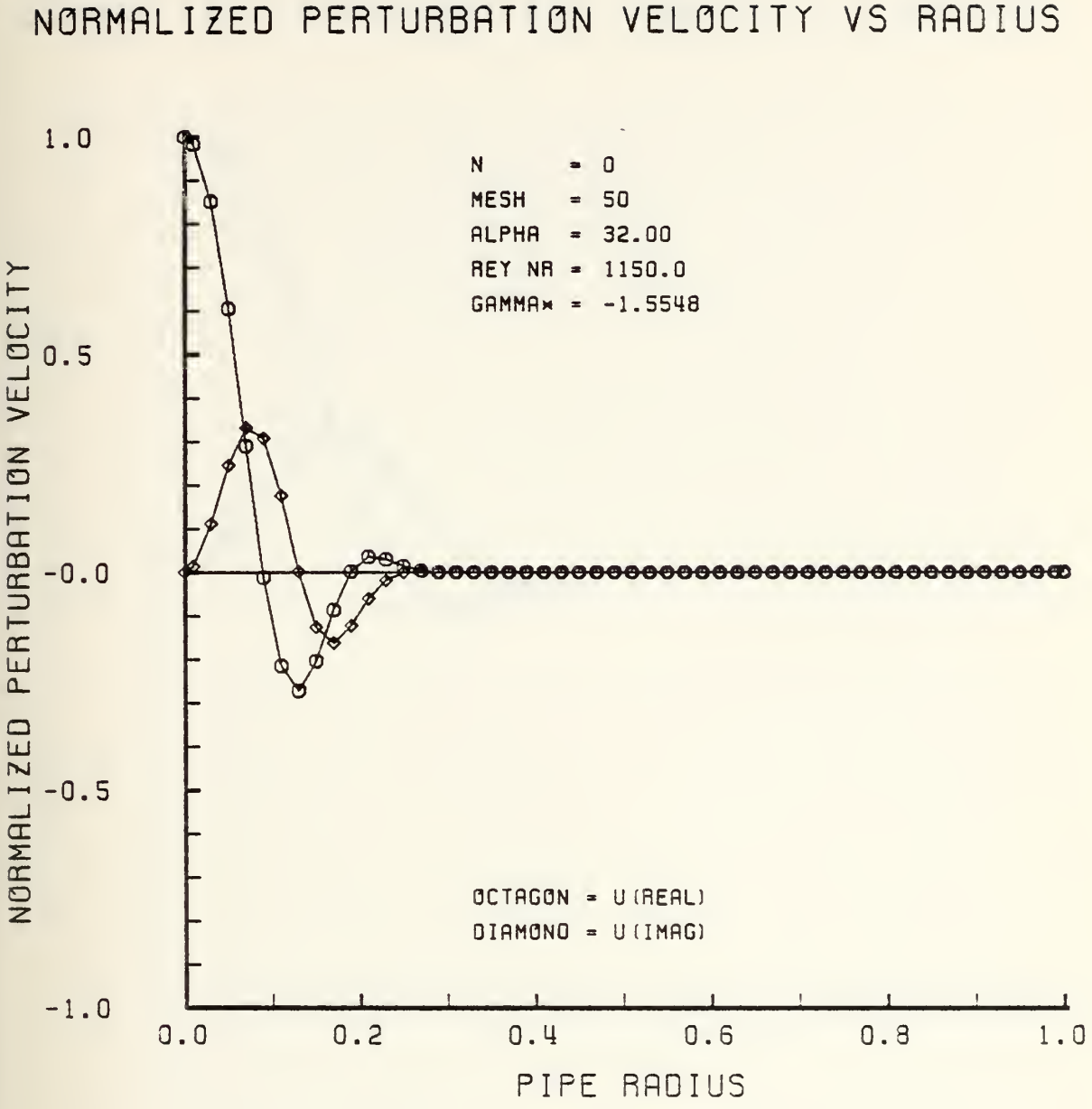

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 8.00
REY NR = 1150.0
GAMMAx = -0.3891
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-5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MESH = 50
ALPHA = 32.00
REY NR = 1150.0
GAMMA× = -1.5548
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-6

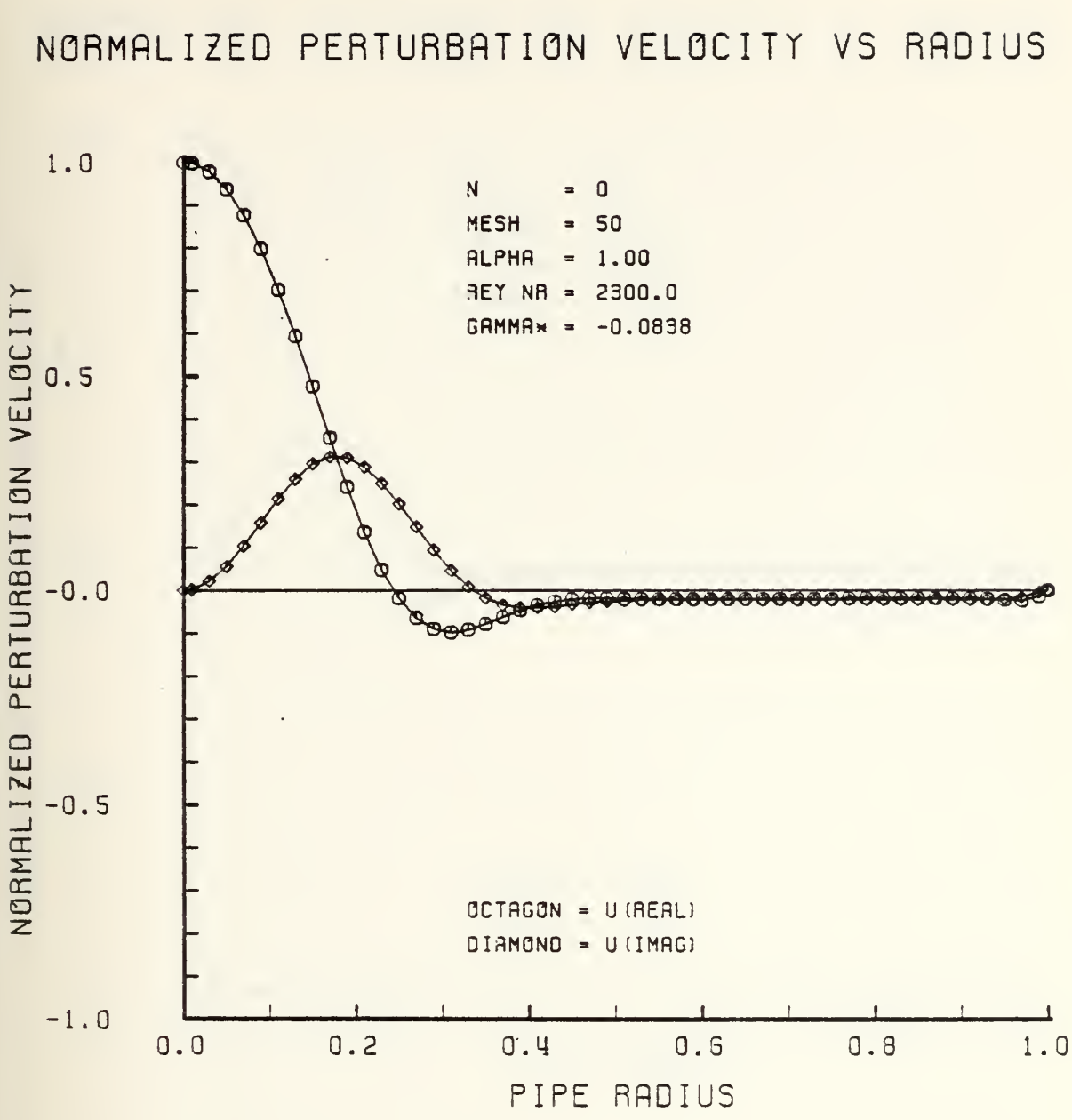
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 2300.0
GAMMAx = -0.0838
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-7

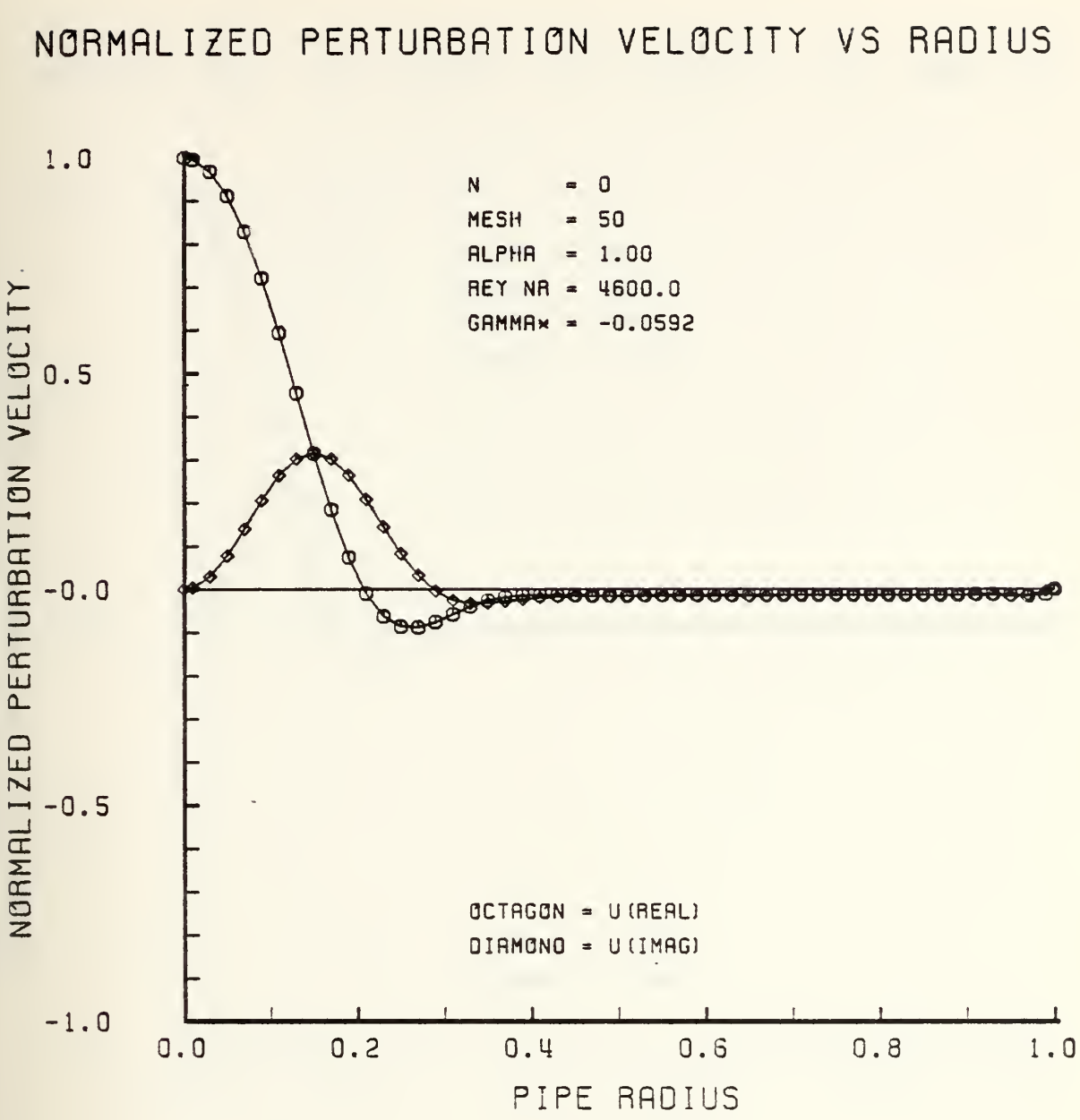
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 4600.0
GAMMA× = -0.0592
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-8

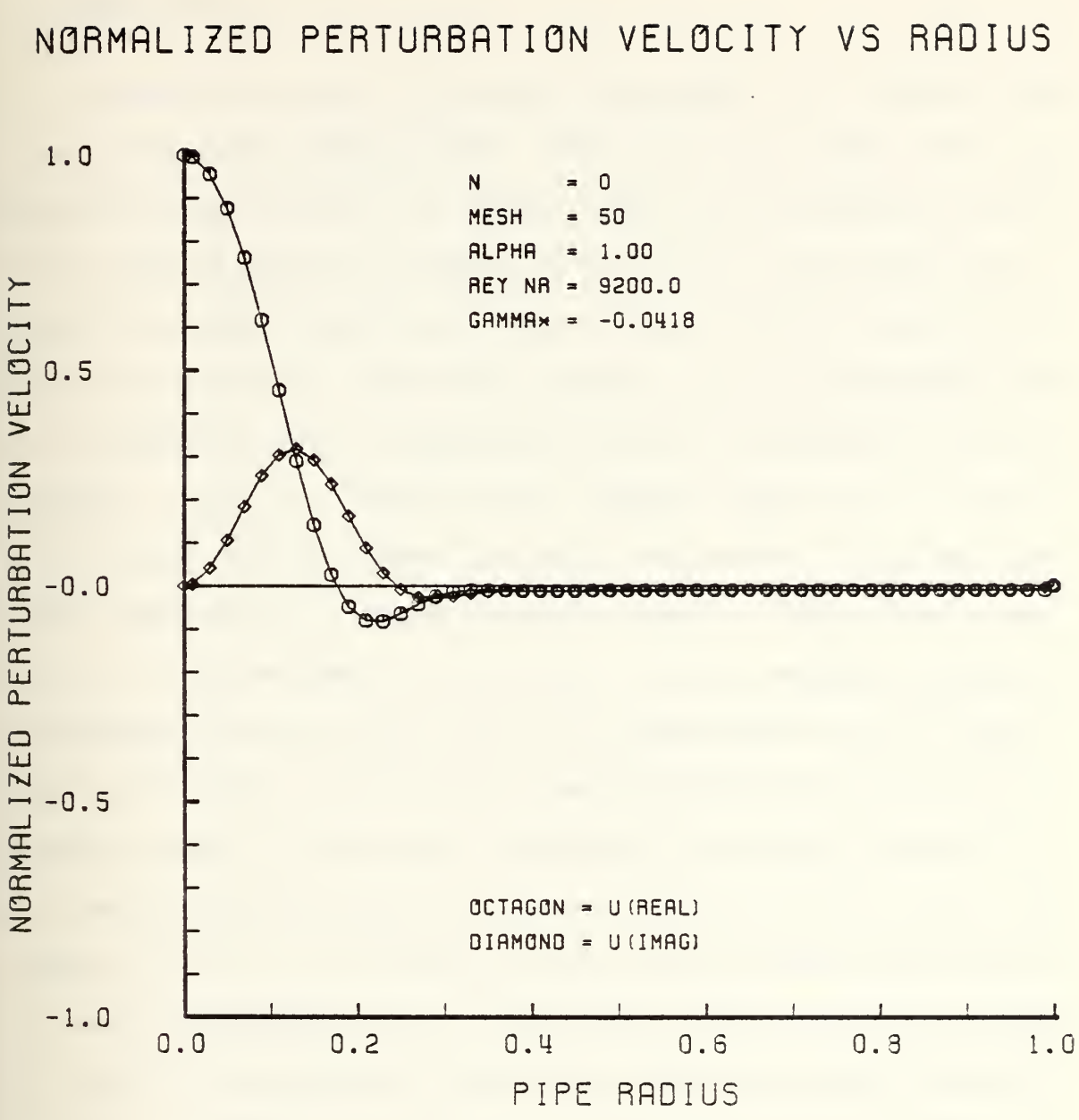
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 9200.0
GAMMAx = -0.0418
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-9

B. RESULTS FOR n = 1

The numerical results for angular wave number n = 1 indicate that flow instabilities indeed exist. There are four factors that may possibly account for the flow instabilities; (1) the complicated nature of the coupled governing equations for n = 1, (2) the rigorous way in which the boundary conditions at the axis were derived and enforced, (3) the introduction of additional parameters into the eigensystem, and (4) the appearance of the eigenvalue in a pair of the boundary condition equations. Table 6-2 summarizes the stability data for n = 1 over a wide range of Reynolds numbers and axial wave numbers. Once again the dashed table entries indicate inaccurate results. Notice that a mesh size of n = 48 was used for this case. This was necessary because of programming constraints that required the eigensystem to be less than or equal to 100 X 100. It can be seen that flow instabilities exist at all Reynolds numbers. This occurs coincidentally with small values of α. More detailed stability data is presented in Table 6-3 for axial wave numbers $0 \leqq \alpha \leqq 1$. It is clear here that the flow becomes more unstable for very small values of $\alpha < 1$. The stability data presented in Tables 6-2 and 6-3 is more easily interpreted as Reynolds number contours in the plots of GAMMA* vs. alpha. Figure 6-10 shows Reynolds number contours over a large range of alpha. It can be clearly seen that the flow is unstable at all Reynolds numbers for small values of axial wave numbers. Figure 6-11 shows a more detailed plot of the Reynolds number contours over a small range of α. Here it can be seen that the flow becomes more unstable for decreasing values of Reynolds numbers! It can

Table 6-2. Stability Data for Angular Wave Number n = 1

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=1.0	α=2.0	α=4.0	α=8.0	α=16.0	α=32.0
300	0.0606	0.0626	-0.0215	-0.2883	-0.3734	-0.7623	-1.3222	-
575	0.0316	0.0270	-0.0601	-0.1983	-0.2530	-0.4638	-1.1246	-0.7546
1150	0.0158	0.0026	-0.0876	-0.1362	-0.1723	-0.2805	-0.7067	-
2300	0.0079	-0.0131	-0.0675	-0.0947	-0.1205	-0.1745	-0.4004	-
4600	0.0039	-0.0237	-0.0476	-0.0664	-0.0860	-0.1117	-0.2327	-
9200	0.0020	-0.0239	-0.0336	-0.0467	-0.0617	-0.0736	-	-
18400	0.0010	-0.0169	-0.0238	-0.0328	-0.0436	-0.0497	-	-

Note: Values are based on 48 mesh points.

Table 6-3. Stability Data for Angular Wave Number n = 1

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=0.10	α=0.15	α=0.20	α=0.50	α=0.75	α=1.0
300	0.0606	0.0657	0.0718	0.0743	0.0743	0.0626	0.0322	-0.0215
575	0.0316	0.0374	0.0394	0.0395	0.0398	0.0270	-0.0057	-0.0601
1150	0.0158	0.0198	0.0204	0.0208	0.0202	0.0026	-0.0324	-0.0876
2300	0.0079	0.0103	0.0105	0.0096	0.0081	-0.0131	-0.0507	-0.0675
4600	0.0039	0.0053	0.0044	0.0028	0.0006	-0.0237	-0.0413	-0.0476
9200	0.0020	0.0022	0.0006	-0.0016	-0.0042	-0.0239	-0.0292	-0.0336
18400	0.0010	0.0003	-0.0018	-0.0044	-0.0074	-0.0169	-0.0206	-0.0238

Note: Values are based on 48 mesh points.

also be seen here that the flow becomes even more unstable for small values of α only slightly larger than zero. The reason for these results cannot be readily explained but it is obvious that they are in direct conflict with observed experimental results. When $\alpha > 1$, however, the contours show an asymptotic trend toward neutral stability for increasing Reynolds numbers; in other words, increasing Reynolds number has a destabilizing effect on the flow which agrees with experimental results. A slightly different method of presenting the data can be seen in Figure 6-12, as alpha contours in the plot of GAMMA* vs. Reynolds number. This plot further emphasizes the results discussed above; the flow is unstable at all Reynolds numbers for small alpha, increasing Reynolds number has a destabilizing effect on the flow and increasing axial wave number has a stabilizing effect.

Normalized perturbation velocity plots were also produced for this case and show some interesting trends. The behavior of the axial perturbation velocity for selected values of Re and α appear in Figures 6-13 through 6-23 on the following pages. Upon close examination of the perturbation velocity plots, it becomes apparent that the computational mesh is not sufficiently fine to adequately approximate the velocity function as α is increased. This problem will be discussed shortly.

It is interesting to note that the perturbation activity shifts from the pipe wall to the axis for increasing values of the axial wave number while Reynolds number remains constant. At small values of α, the activity originates at the wall. At larger values of α, the activity shifts to the axis, for the most part. At even larger values of α, the activity moves away from the axis to the region between the axis and

wall. However, some degree of activity remains at the wall for the last two cases. The change in the perturbation activity as described above can be seen in Figures 6-14 through 6-16 for Re = 575 and in Figures 6-19 through 6-21 for Re = 2300. The shift of perturbation activity to the axis with increasing axial wave number deserves comment here. Schlichting [Ref. 16] observed that the transition to turbulence is "characterized by the ..... appearance of self-sustaining turbulent flashes with emanate from fluid layers near the wall along the tube." Therefore, it appears that unstable flows are associated with perturbation activity near the wall. As the activity shifts away from the wall with increasing axial wave number, the flow becomes stable. Additionally, after the activity moves to the axis, some degree of activity remains at the wall. While the solution appears to remain stable, it is recommended that this peculiarity be examined further. The use of a finer mesh to resolve this activity at the wall would be useful here.

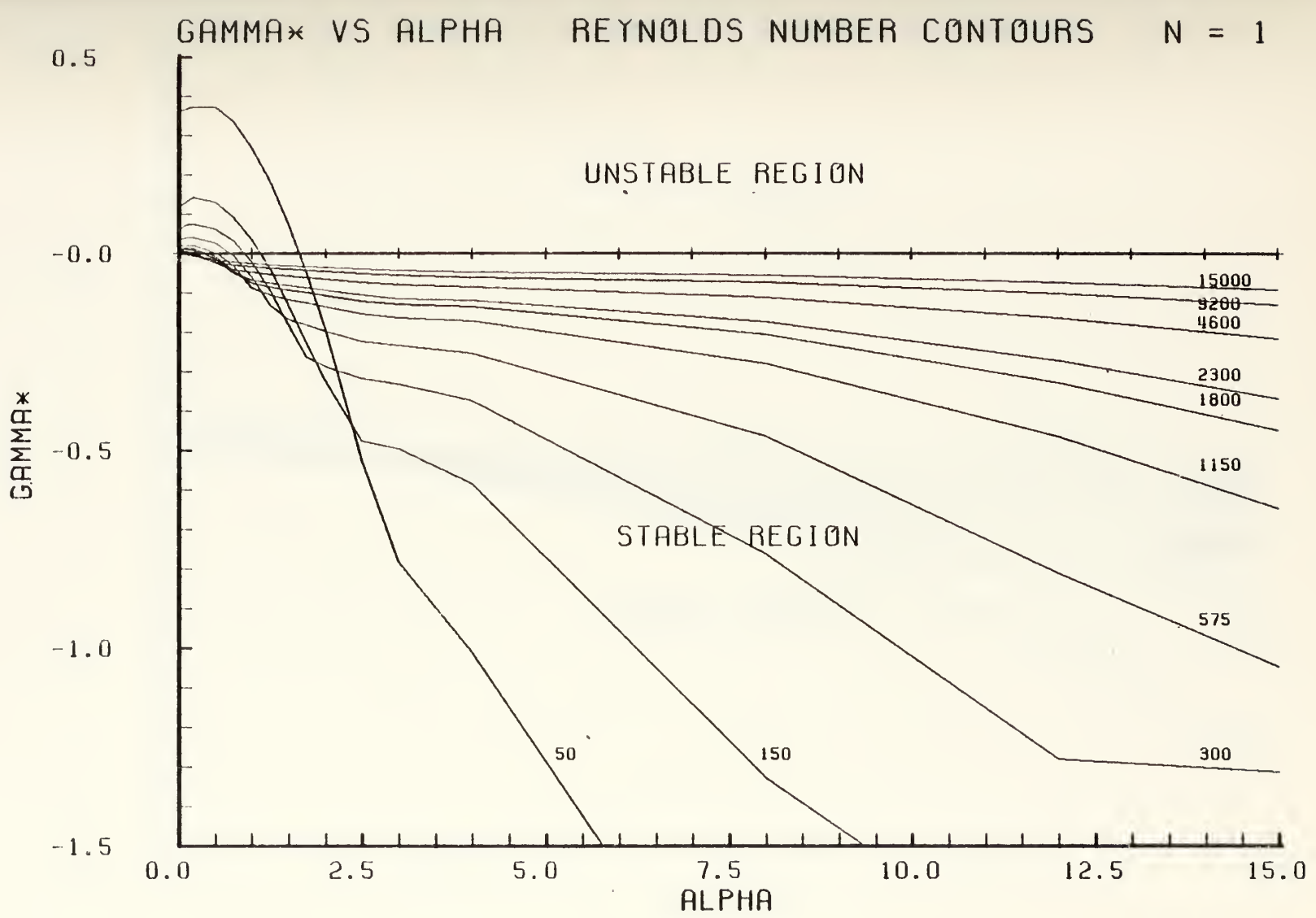
GAMMA* VS ALPHA REYNOLDS NUMBER CONTOURS N = 1
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
575
300
150
50
GAMMA*
ALPHA
0.5
-0.0
-0.5
-1.0
-1.5
0.0
2.5
5.0
7.5
10.0
12.5
15.0

Figure 6-10

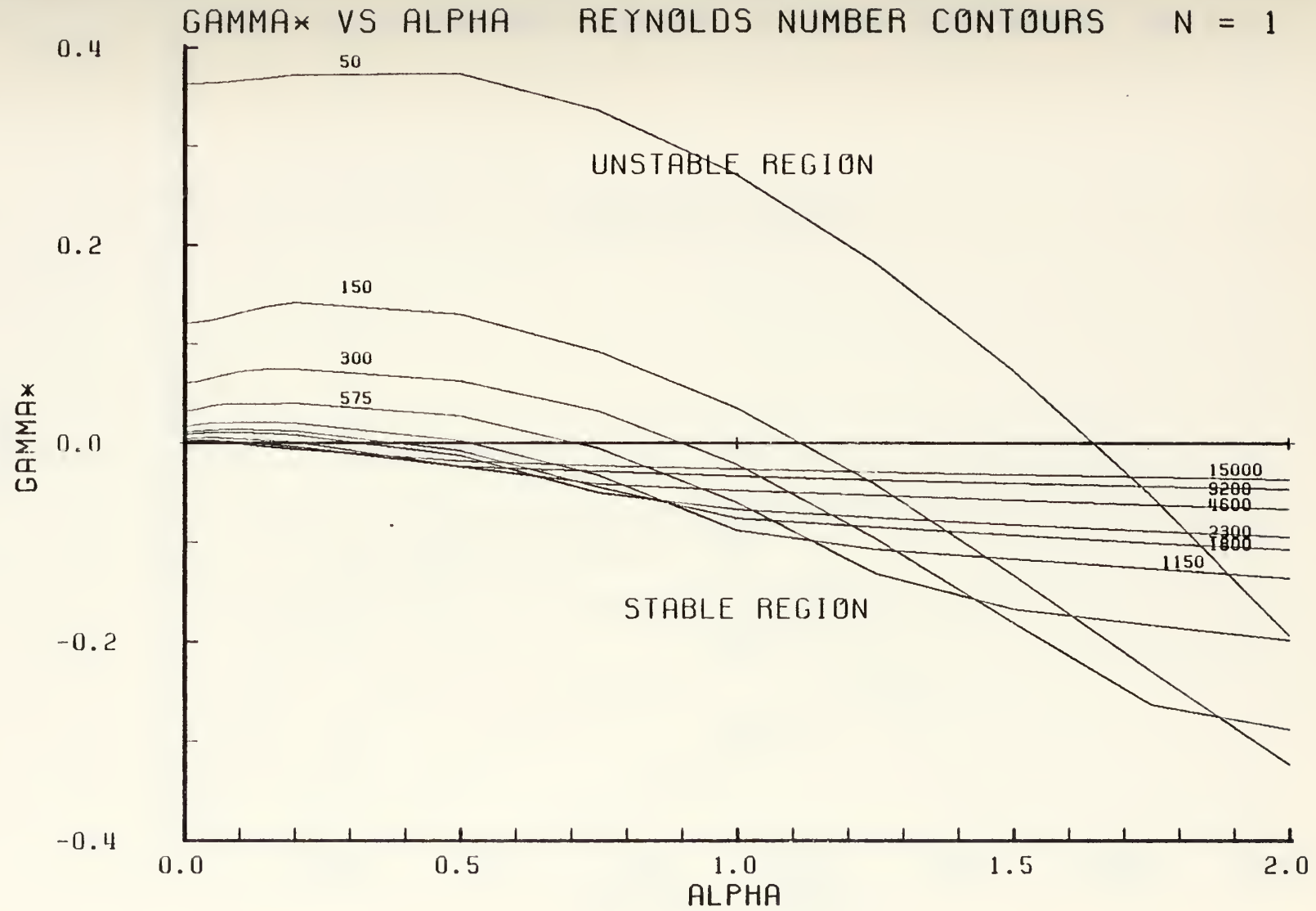
GAMMA* VS ALPHA REYNOLDS NUMBER CONTOURS N = 1
0.4
0.2
0.0
-0.2
-0.4
GAMMA*
50
150
300
575
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
0.0
0.5
1.0
1.5
2.0
ALPHA

Figure 6-11

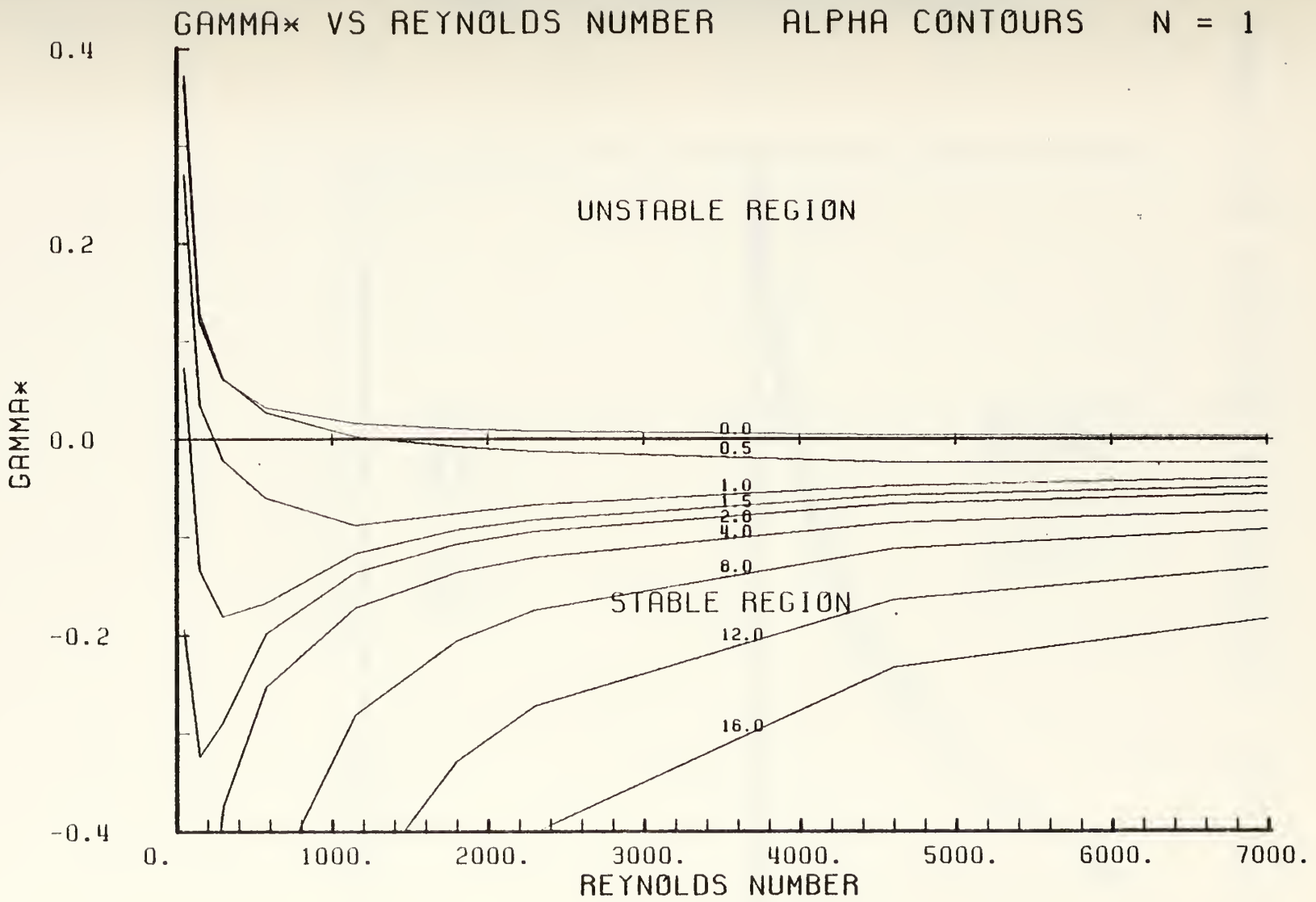
GAMMA* VS REYNOLDS NUMBER ALPHA CONTOURS N = 1
UNSTABLE REGION
STABLE REGION
0.0
0.5
1.0
1.5
2.0
4.0
8.0
12.0
16.0
0.4
0.2
0.0
-0.2
-0.4
GAMMA*
0.
1000.
2000.
3000.
4000.
5000.
6000.
7000.
REYNOLDS NUMBER

Figure 6-12

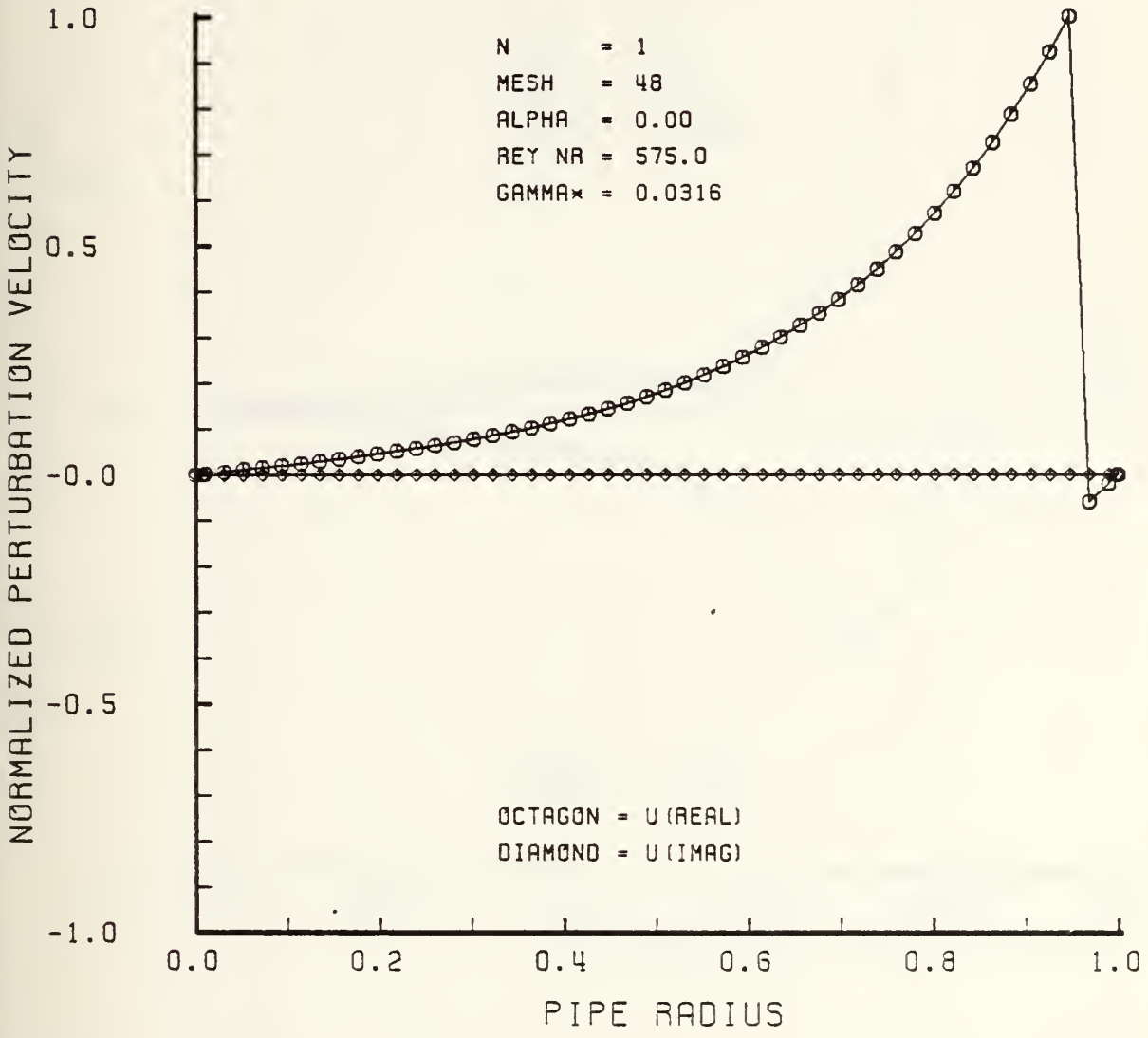
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 0.00
REY NR = 575.0
GAMMA× = 0.0316
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-13

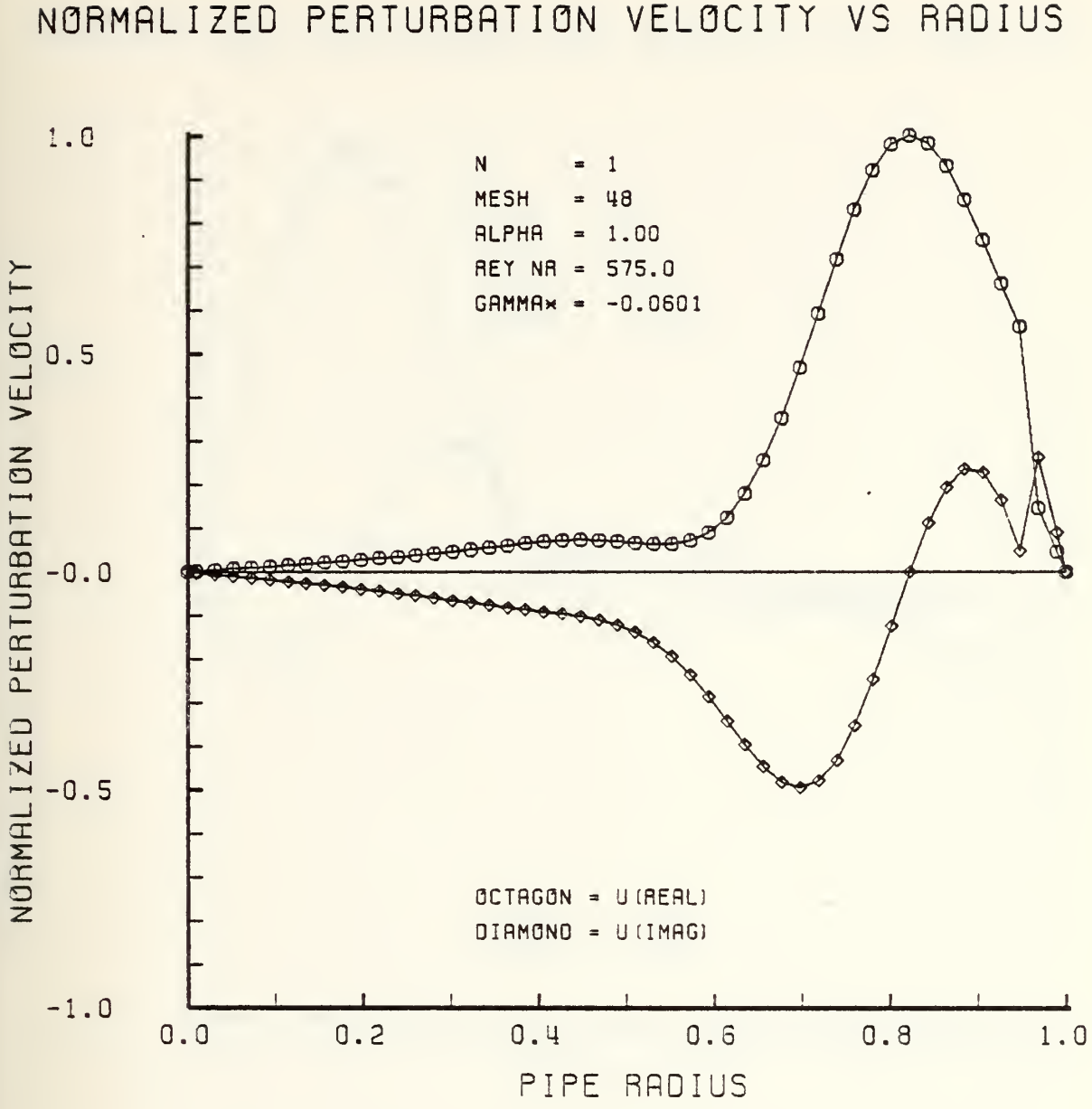
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 1.00
REY NR = 575.0
GAMMAx = -0.0601
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-14

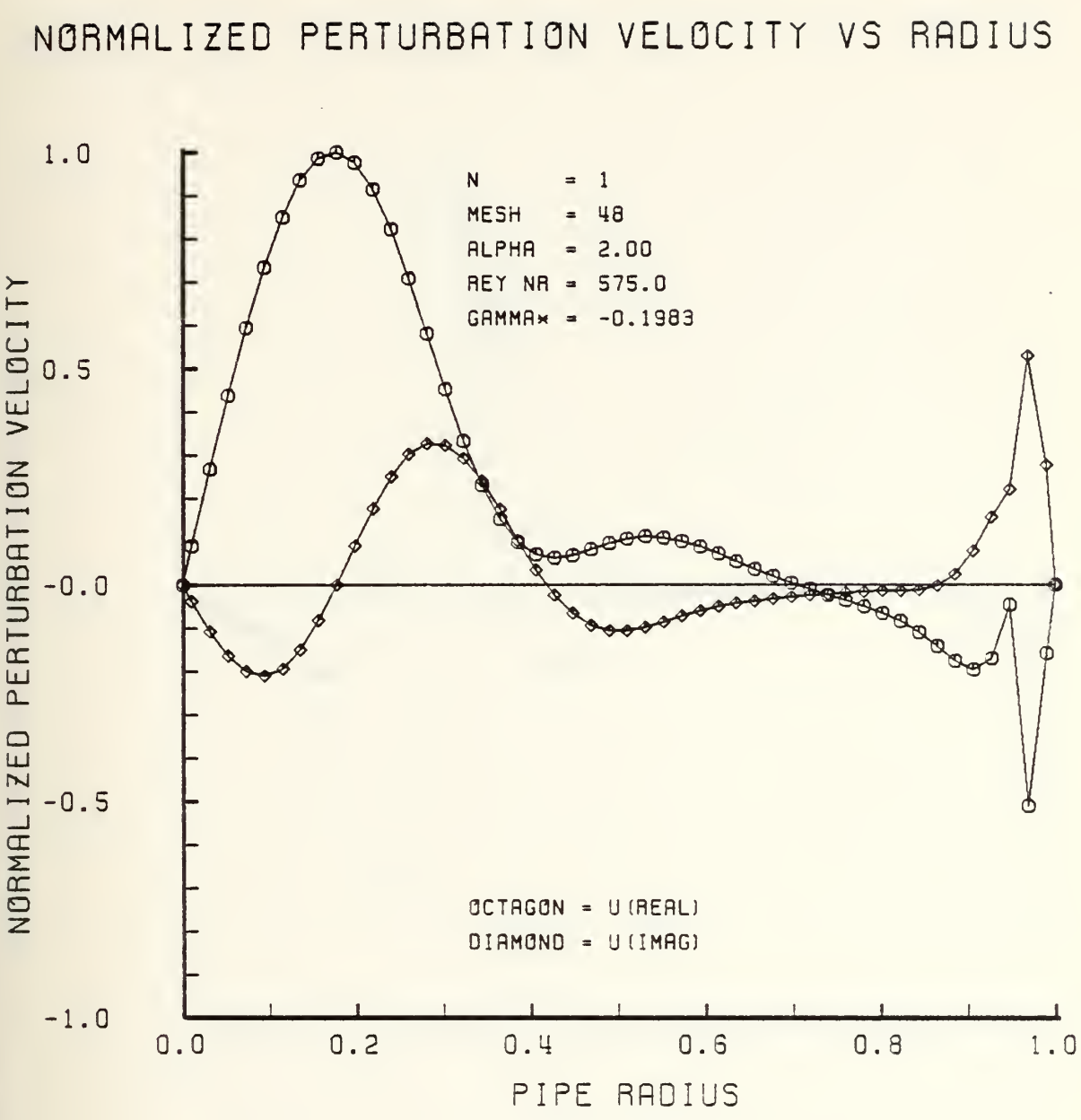
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 2.00
REY NR = 575.0
GAMMA× = -0.1983
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-15

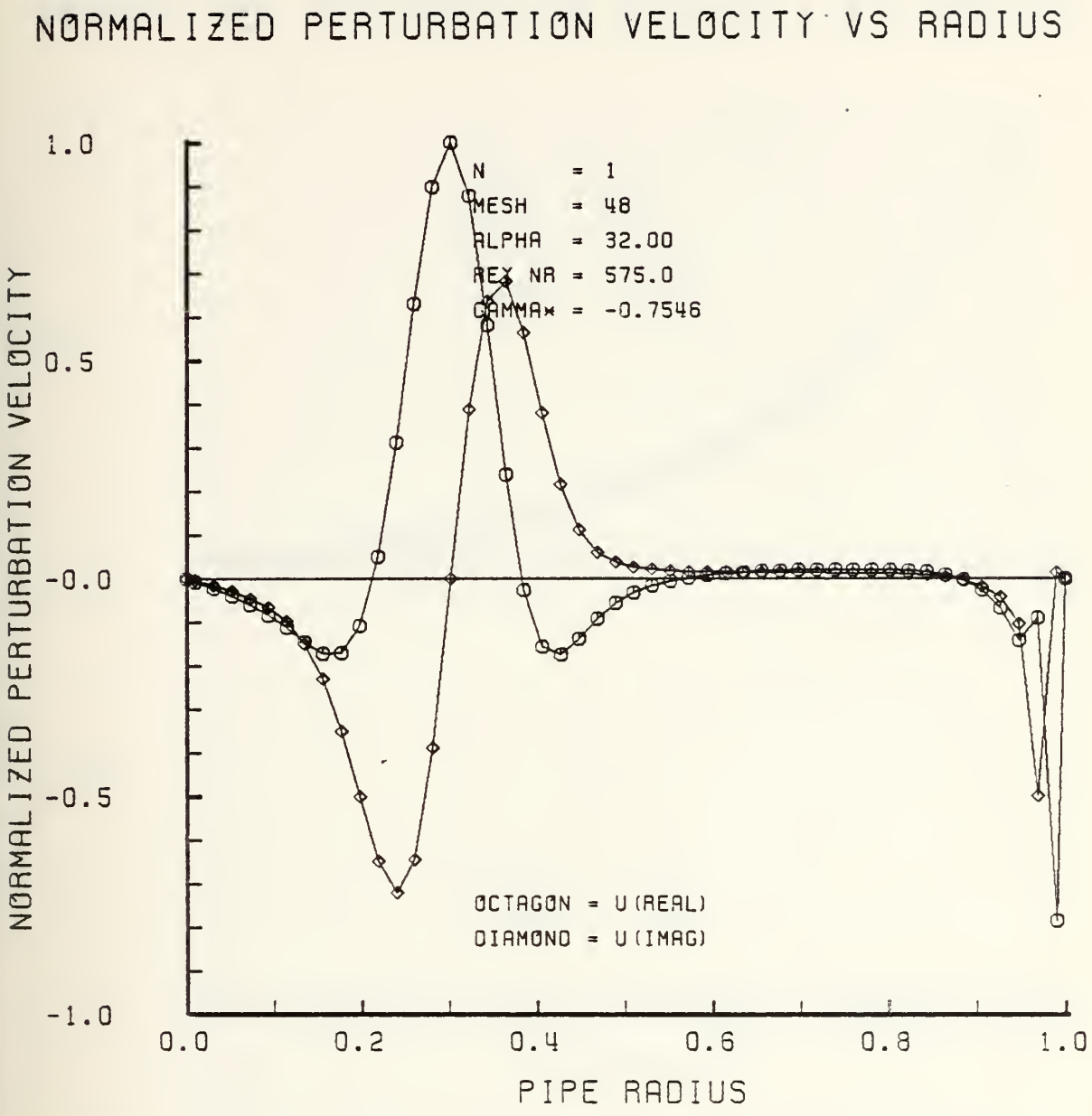
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 32.00
REX NR = 575.0
GAMMA* = -0.7546
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-16

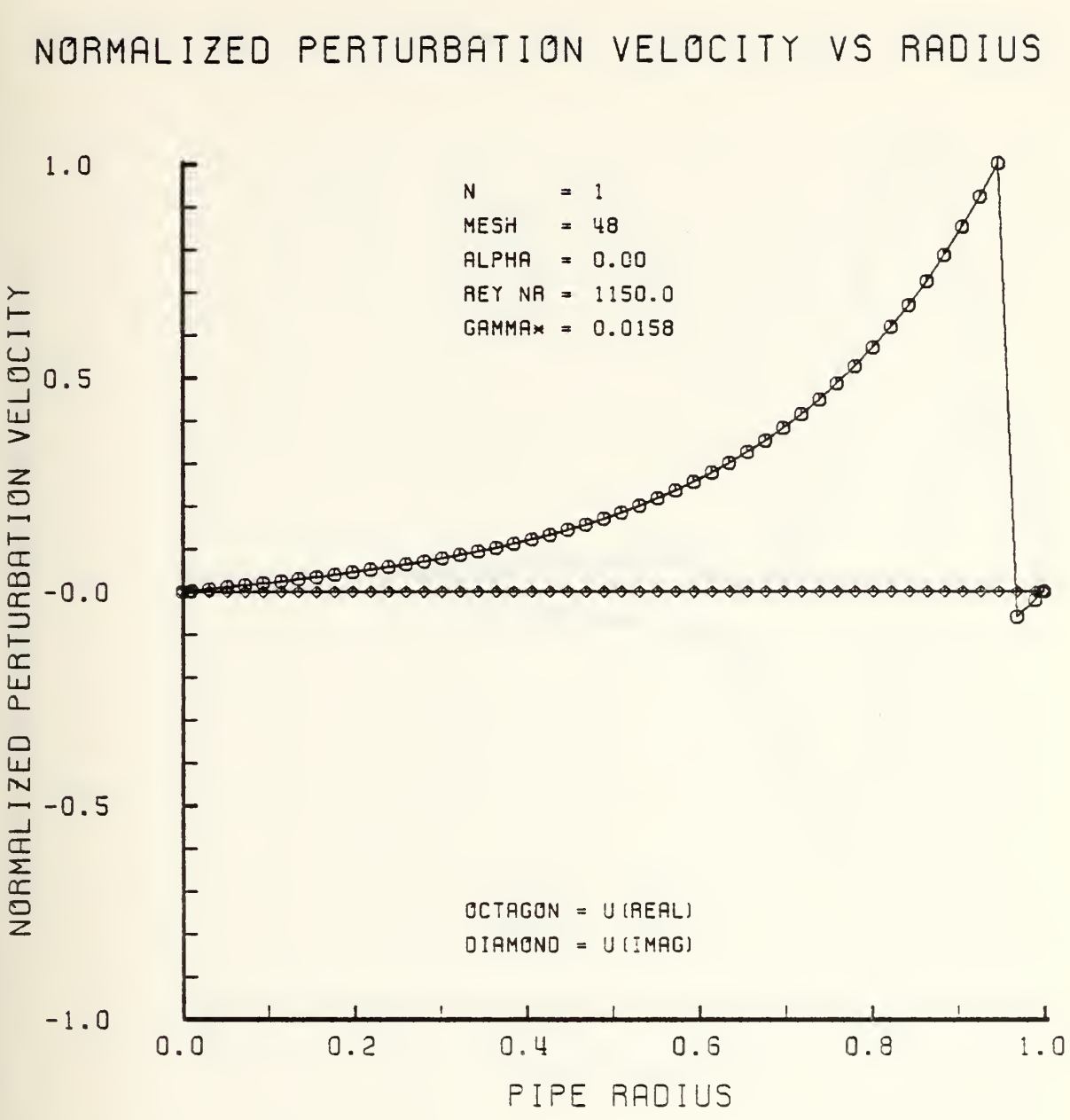
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 0.00
REY NR = 1150.0
GAMMAx = 0.0158
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-17

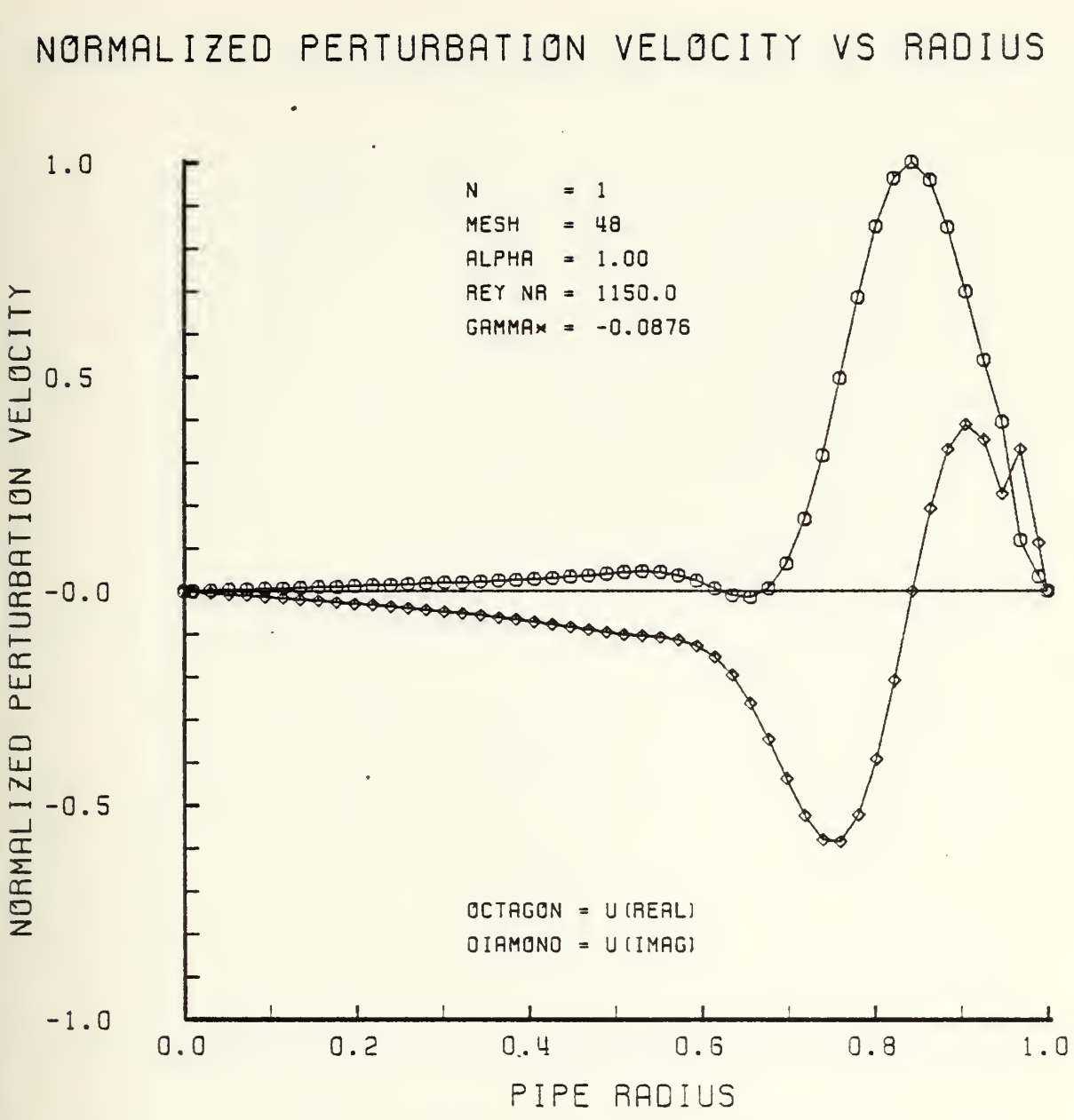
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 1.00
REY NR = 1150.0
GAMMAx = -0.0876
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-18

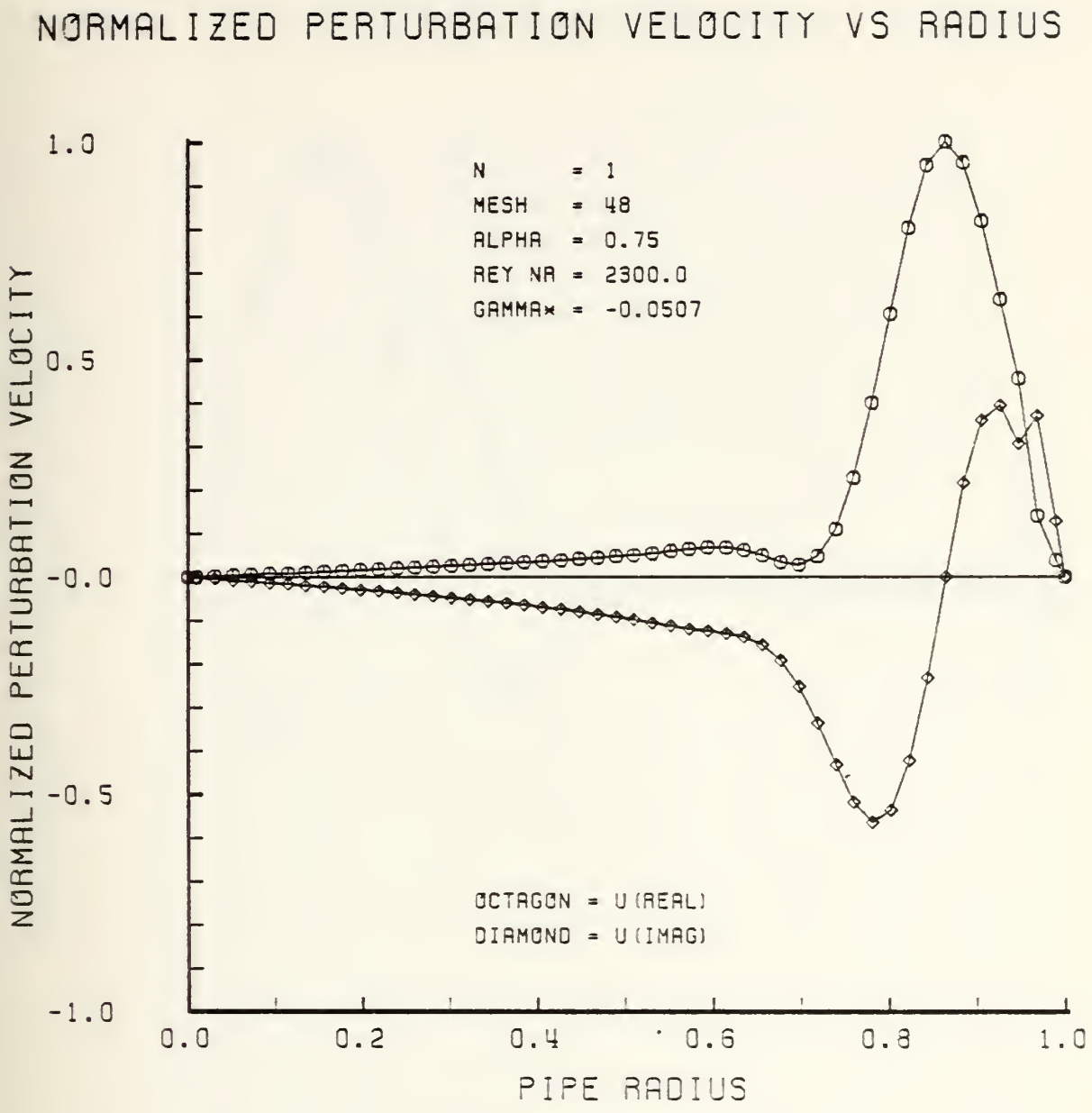
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 0.75
REY NR = 2300.0
GAMMA* = -0.0507
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-19

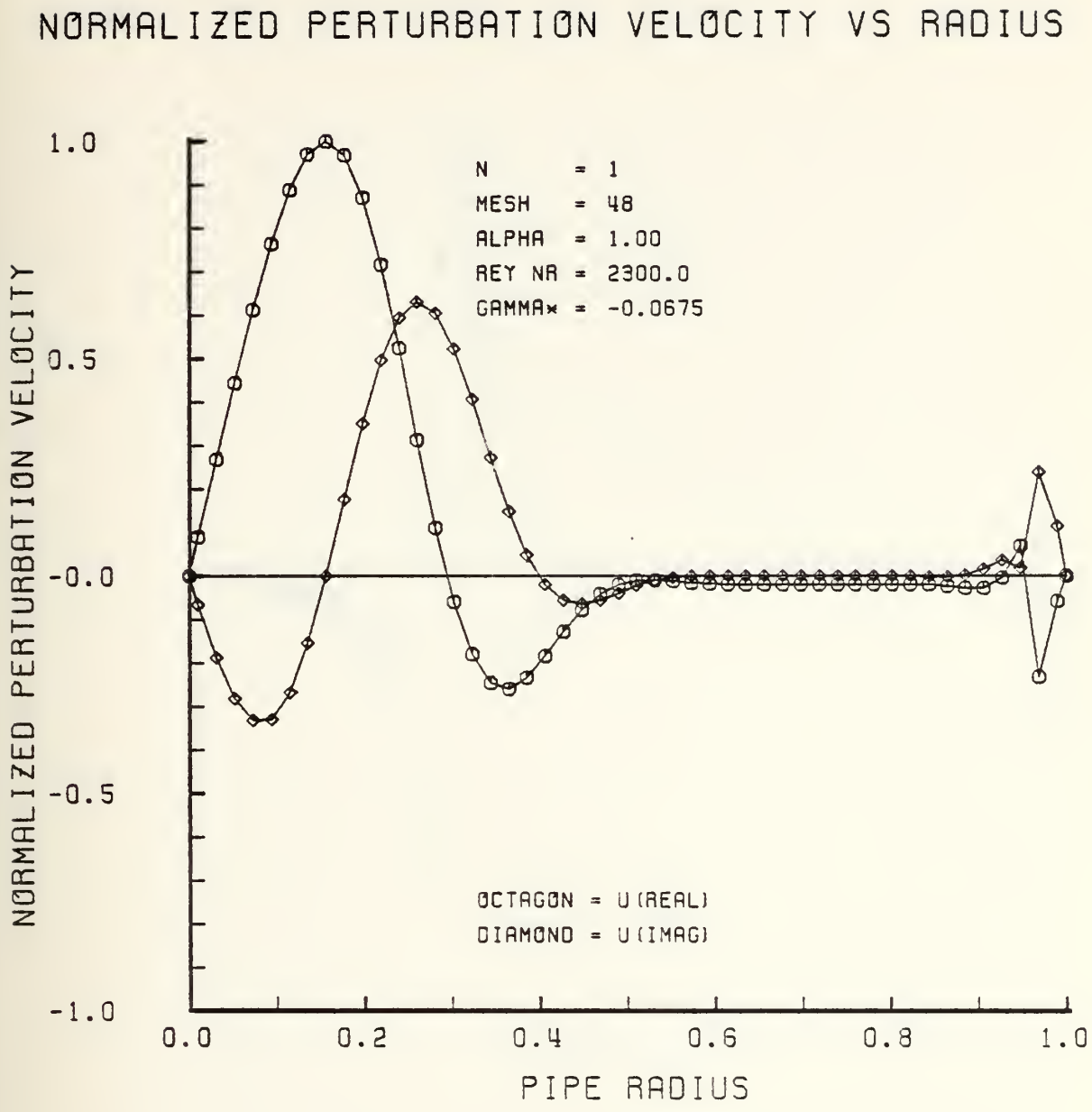
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 1.00
REY NR = 2300.0
GAMMAx = -0.0675
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
PIPE RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

Figure 6-20

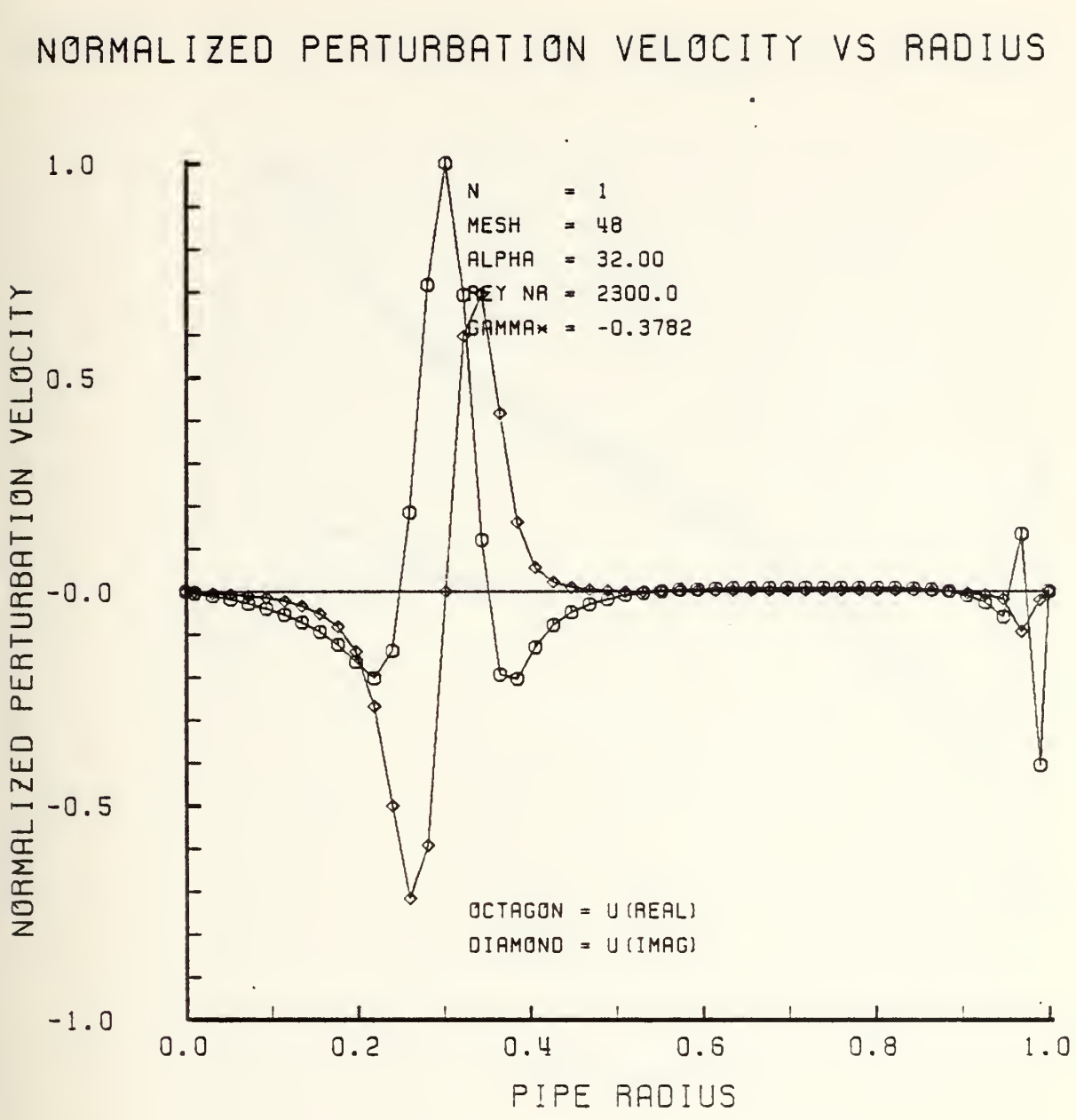
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 32.00
REY NR = 2300.0
GAMMA× = -0.3782
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-21

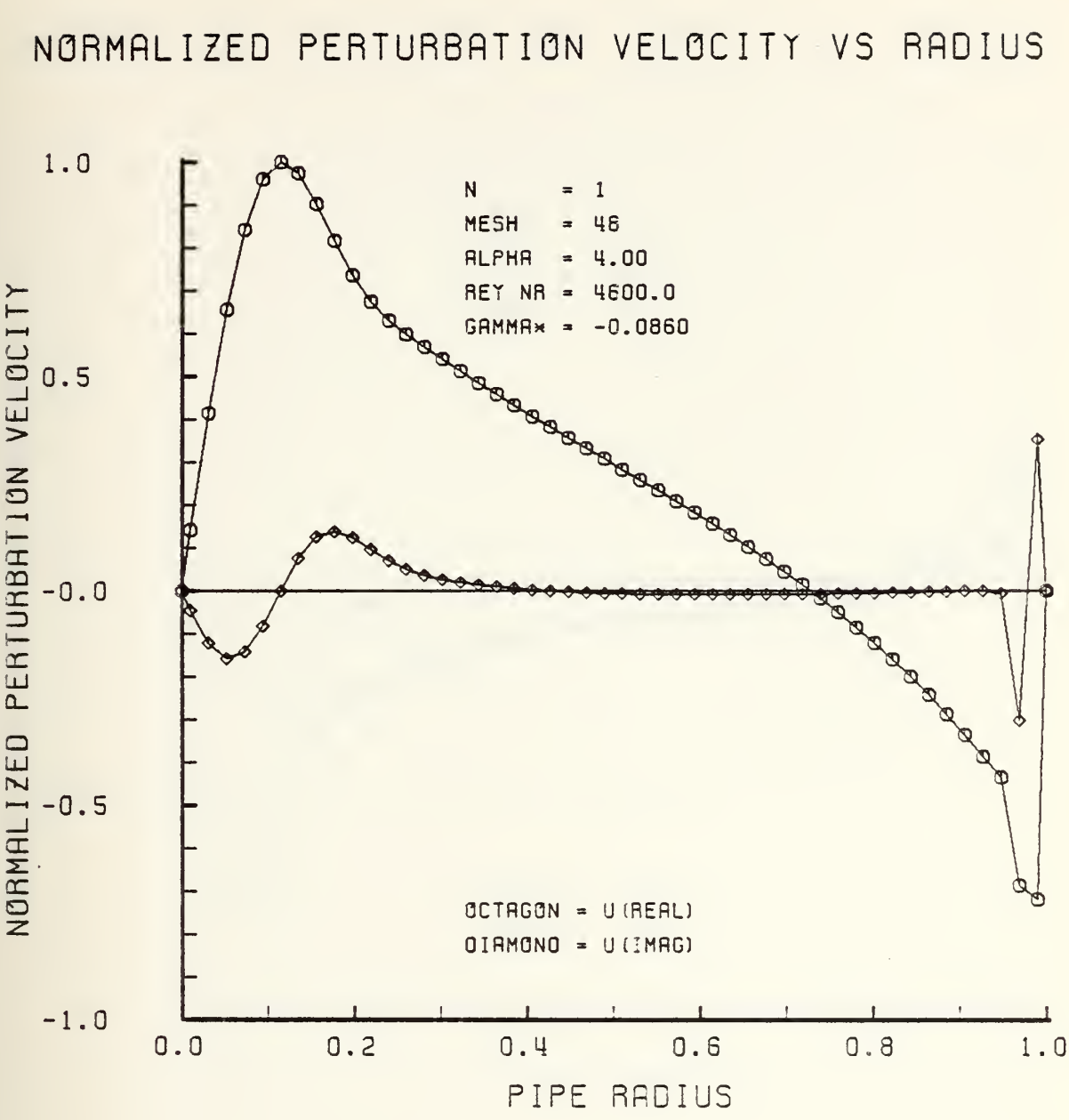
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 4.00
REY NR = 4600.0
GAMMA* = -0.0860
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-22

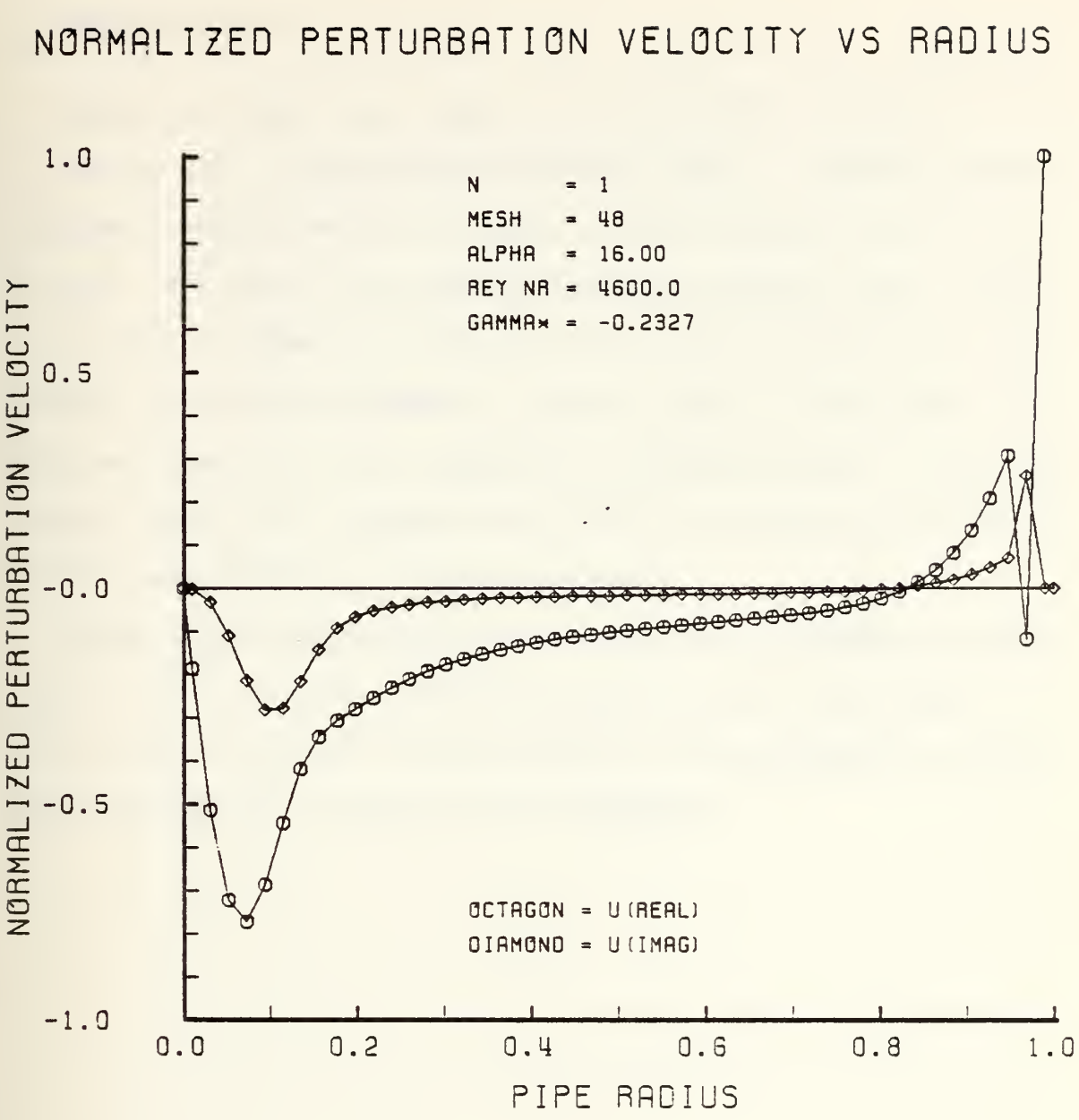
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 16.00
REY NR = 4600.0
GAMMA* = -0.2327
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-23

C. RESULTS FOR n = 6

Results for angular wave number n = 6 are similar to those for n = 0 in that the flow is stable at all Reynolds numbers. Stability data is presented in Tables 6-4 and 6-5 where it can be determined that this is the case. The data is once again presented as Reynolds number contours in the plots of GAMMA* vs. alpha in Figures 6-24 and 6-25 and as alpha contours in the plot of GAMMA* vs. Reynolds number in Figure 6-26. The trends are clear; the flow is stable at all Reynolds numbers, increasing Reynolds number has a destabilizing effect on the flow and increasing the axial wave number has a stabilizing effect.

Figures 6-27 through 6-36 are representative perturbation velocity plots for n = 6 at several Reynolds numbers and axial wave numbers. It can be seen in Figures 6-31 and 6-32 that the perturbation activity shifts from the wall to the axis for increasing α.

Table 6-4. Stability Data for Angular Wave Number n = 6

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=1.0	α=2.0	α=4.0	α=8.0	α=16.0	α=32.0
300	-0.3291	-0.3512	-0.4829	-0.7295	-1.1654	-1.9718	-2.8067	-4.6185
575	-0.1717	-0.2450	-0.3644	-0.5651	-0.9064	-1.4798	-2.0927	-
1150	-0.0858	-0.1814	-0.2789	-0.4386	-0.7049	-1.0532	-	-
2300	-0.0429	-0.1390	-0.2173	-0.3447	-0.5136	-	-	-
4600	-0.0215	-0.1084	-0.1712	-0.2549	-0.3631	-	-	-
9200	-0.0107	-0.0855	-0.1272	-0.1802	-0.2565	-	-	-
18400	-0.0054	-0.0636	-0.0899	-0.1273	-0.1811	-	-	-

Note: Values are based on 50 mesh points.

Table 6-5. Stability Data for Angular Wave Number n = 6

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=0.10	α=0.15	α=0.20	α=0.50	α=0.75	α=1.0
300	-0.3291	-0.3267	-0.3228	-0.3201	-0.3192	-0.3512	-0.4156	-0.4829
575	-0.1717	-0.1686	-0.1665	-0.1679	-0.1727	-0.2450	-0.3071	-0.3644
1150	-0.0858	-0.0833	-0.0864	-0.0960	-0.1092	-0.1814	-0.2325	-0.2789
2300	-0.0429	-0.0432	-0.0546	-0.0675	-0.0794	-0.1390	-0.1801	-0.2173
4600	-0.0215	-0.0273	-0.0397	-0.0505	-0.0603	-0.1084	-0.1414	-0.1712
9200	-0.0107	-0.0198	-0.0302	-0.0390	-0.0469	-0.0855	-0.1102	-0.1272
18400	-0.0054	-0.0151	-0.0234	-0.0305	-0.0368	-0.0636	-0.0779	-0.0899

Note: Values are based on 50 mesh points.

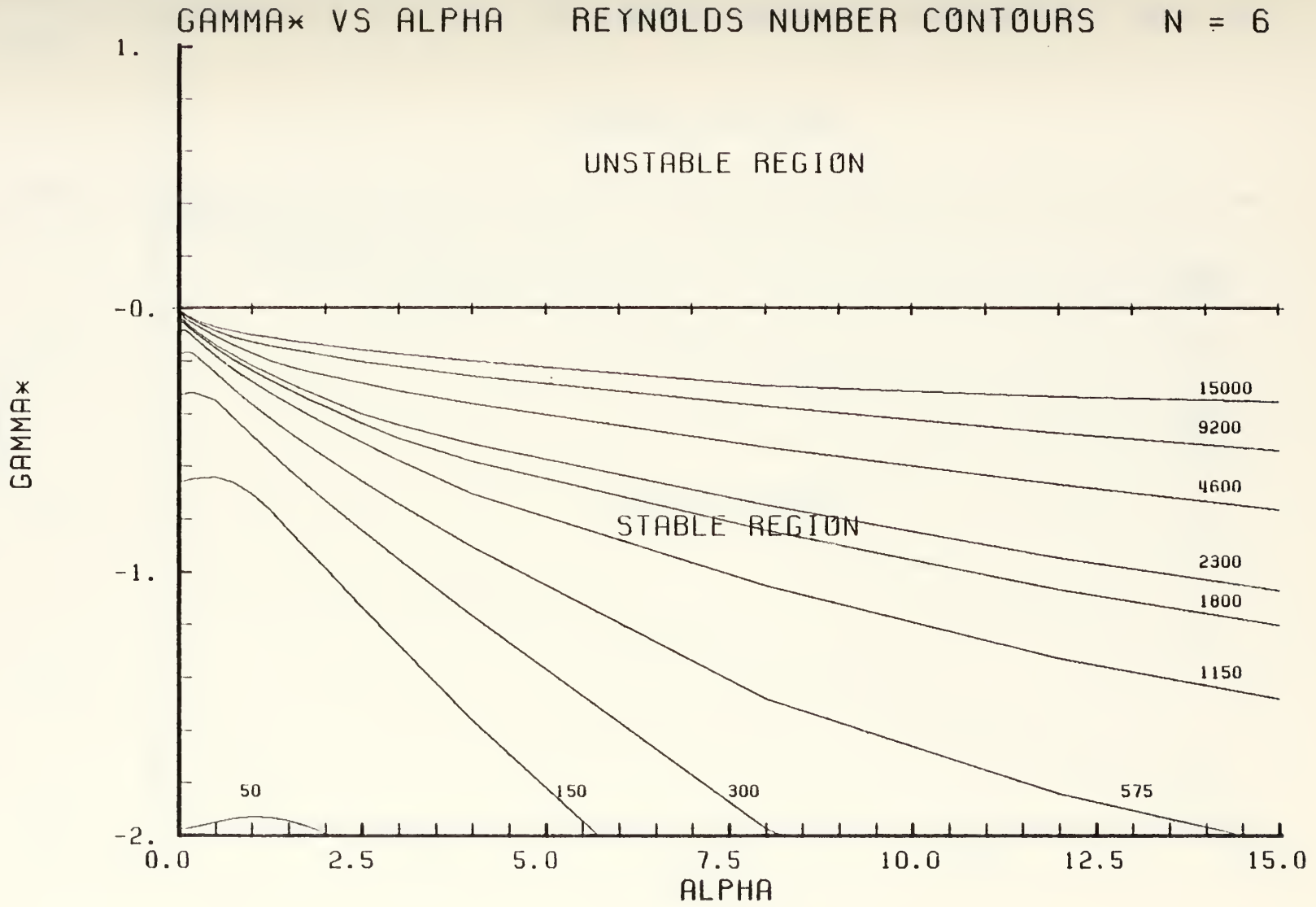
GAMMA* VS ALPHA REYNOLDS NUMBER CONTOURS N = 6
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
575
300
150
50
1.
-0.
-1.
-2.
GAMMA*
0.0
2.5
5.0
7.5
10.0
12.5
15.0
ALPHA

Figure 6-24

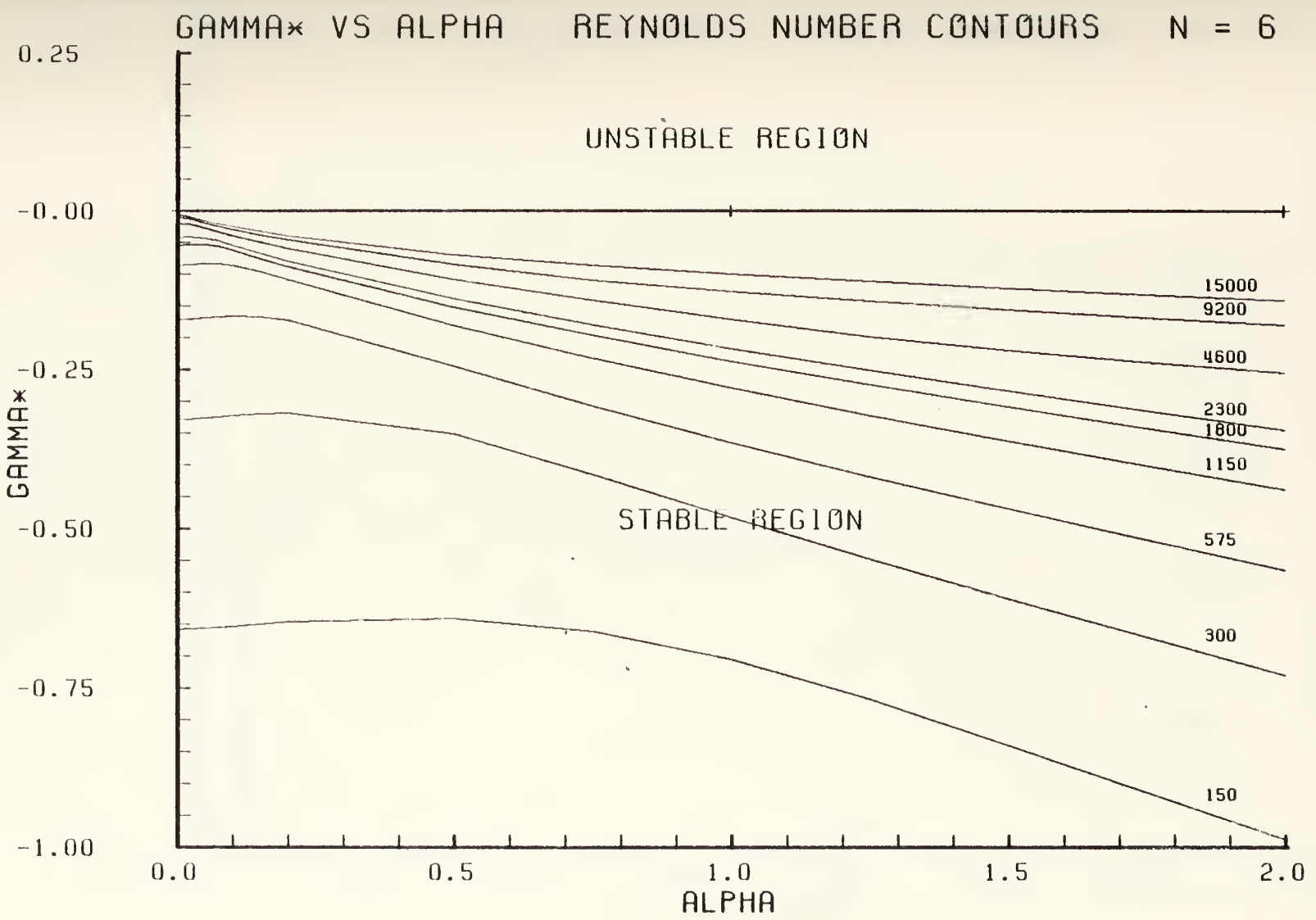
GAMMA* VS ALPHA REYNOLDS NUMBER CONTOURS N = 6
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
575
300
150
0.25
-0.00
-0.25
-0.50
-0.75
-1.00
GAMMA*
0.0
0.5
1.0
1.5
2.0
ALPHA

Figure 6-25

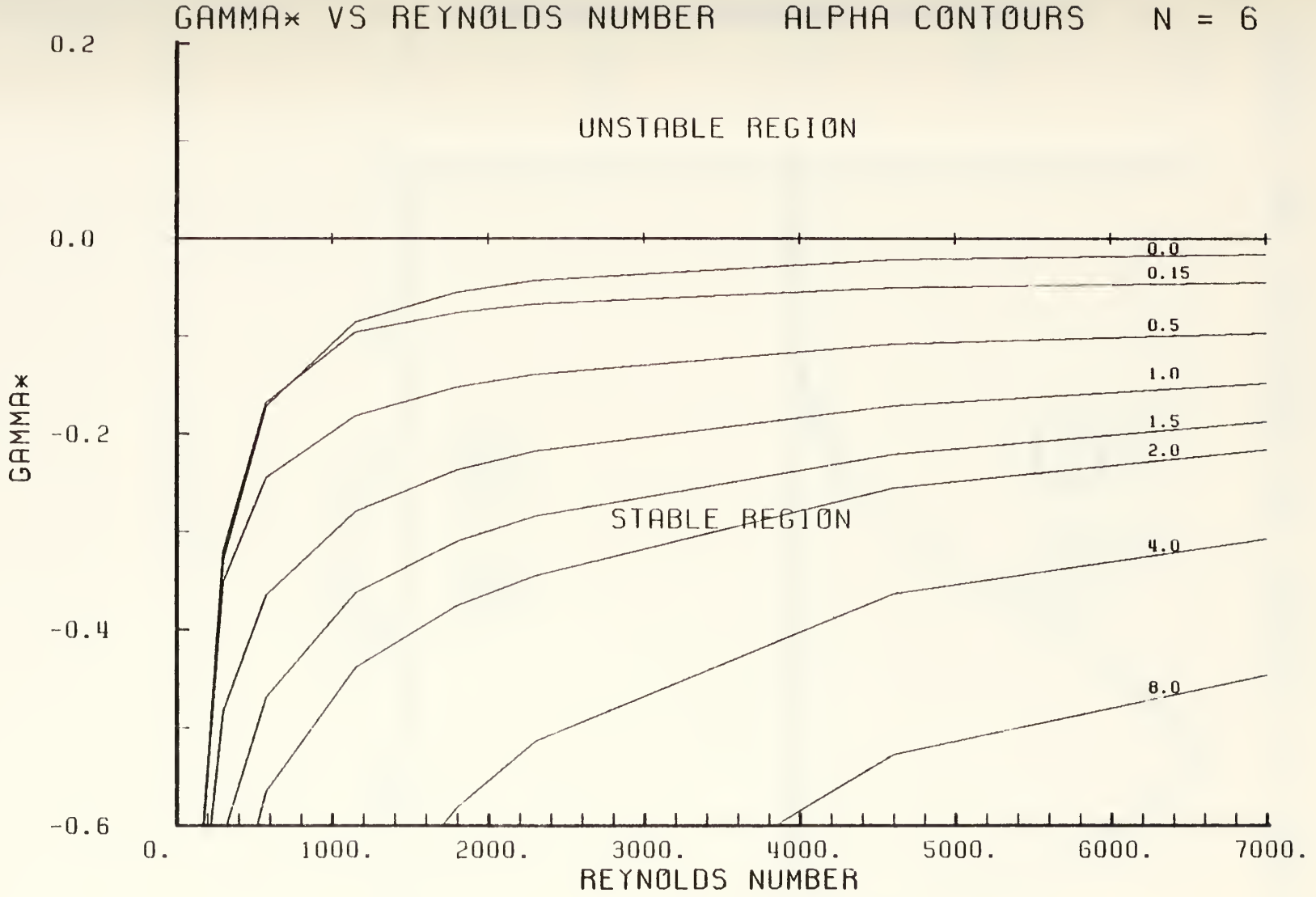
GAMMA* VS REYNOLDS NUMBER ALPHA CONTOURS N = 6
UNSTABLE REGION
STABLE REGION
0.0
0.15
0.5
1.0
1.5
2.0
4.0
8.0
0.2
0.0
-0.2
-0.4
-0.6
GAMMA*
0.
1000.
2000.
3000.
4000.
5000.
6000.
7000.
REYNOLDS NUMBER

Figure 6-26

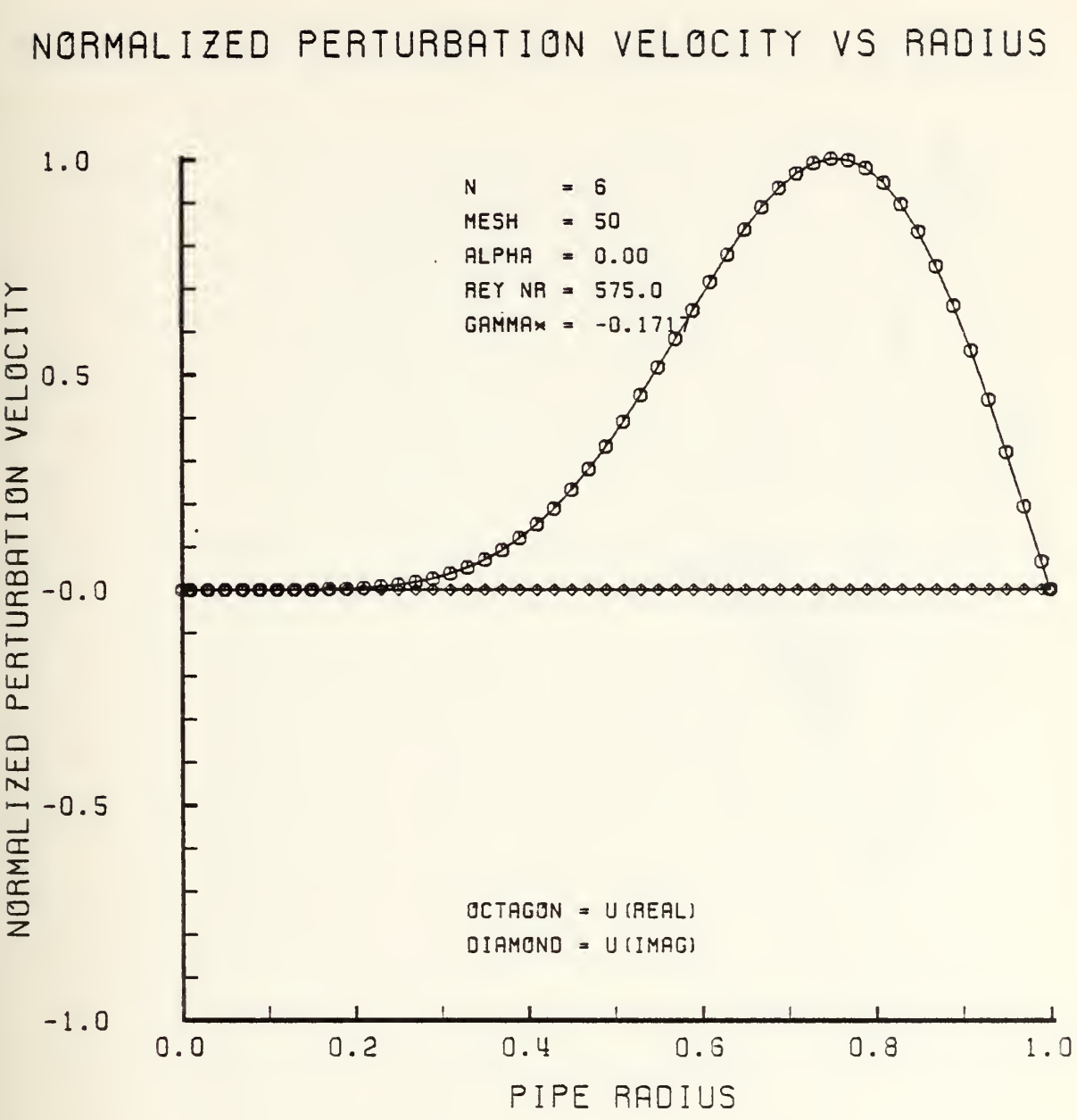
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 0.00
REY NR = 575.0
GAMMAx = -0.171
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-27

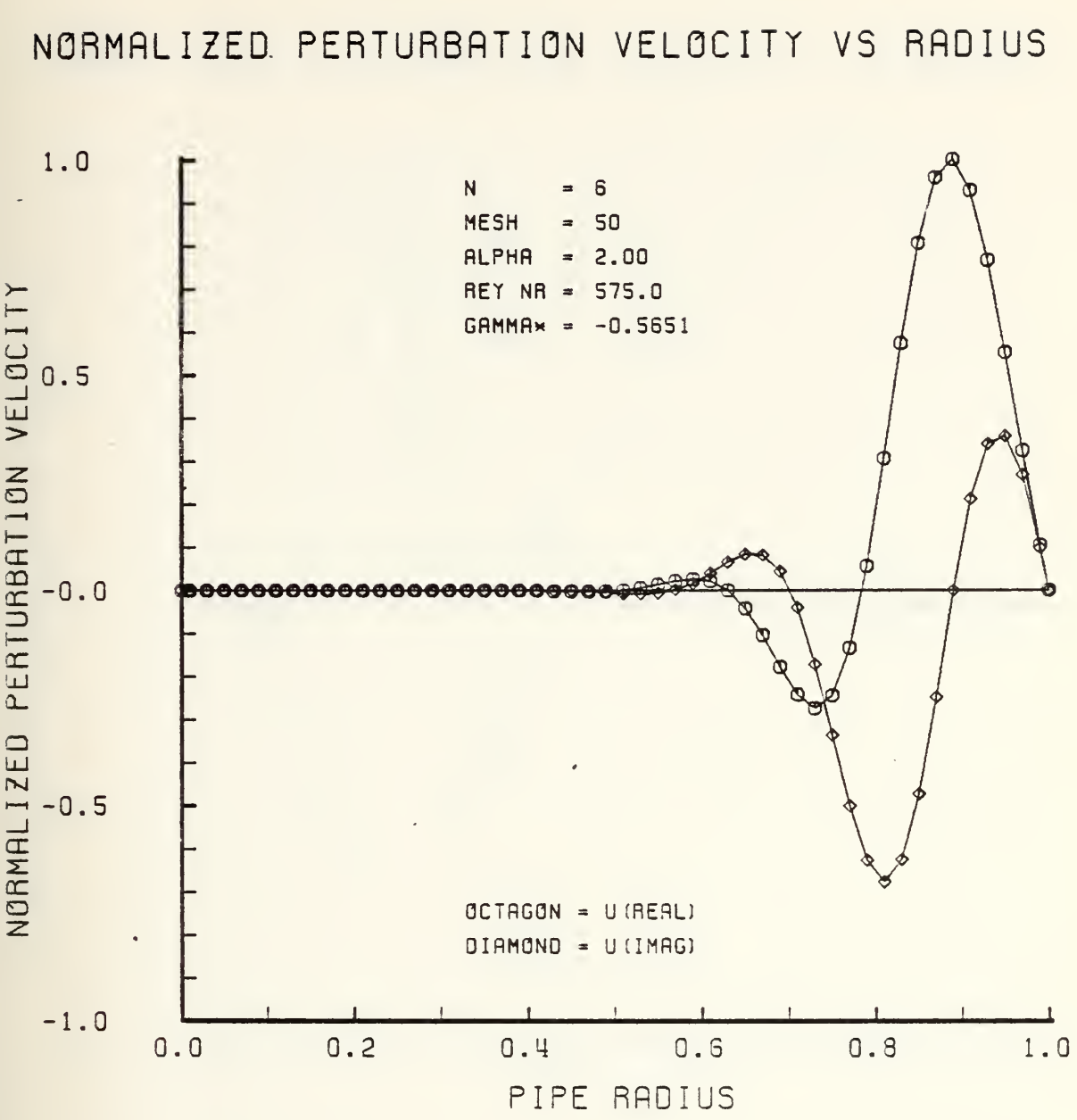
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 2.00
REY NR = 575.0
GAMMAx = -0.5651
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-28

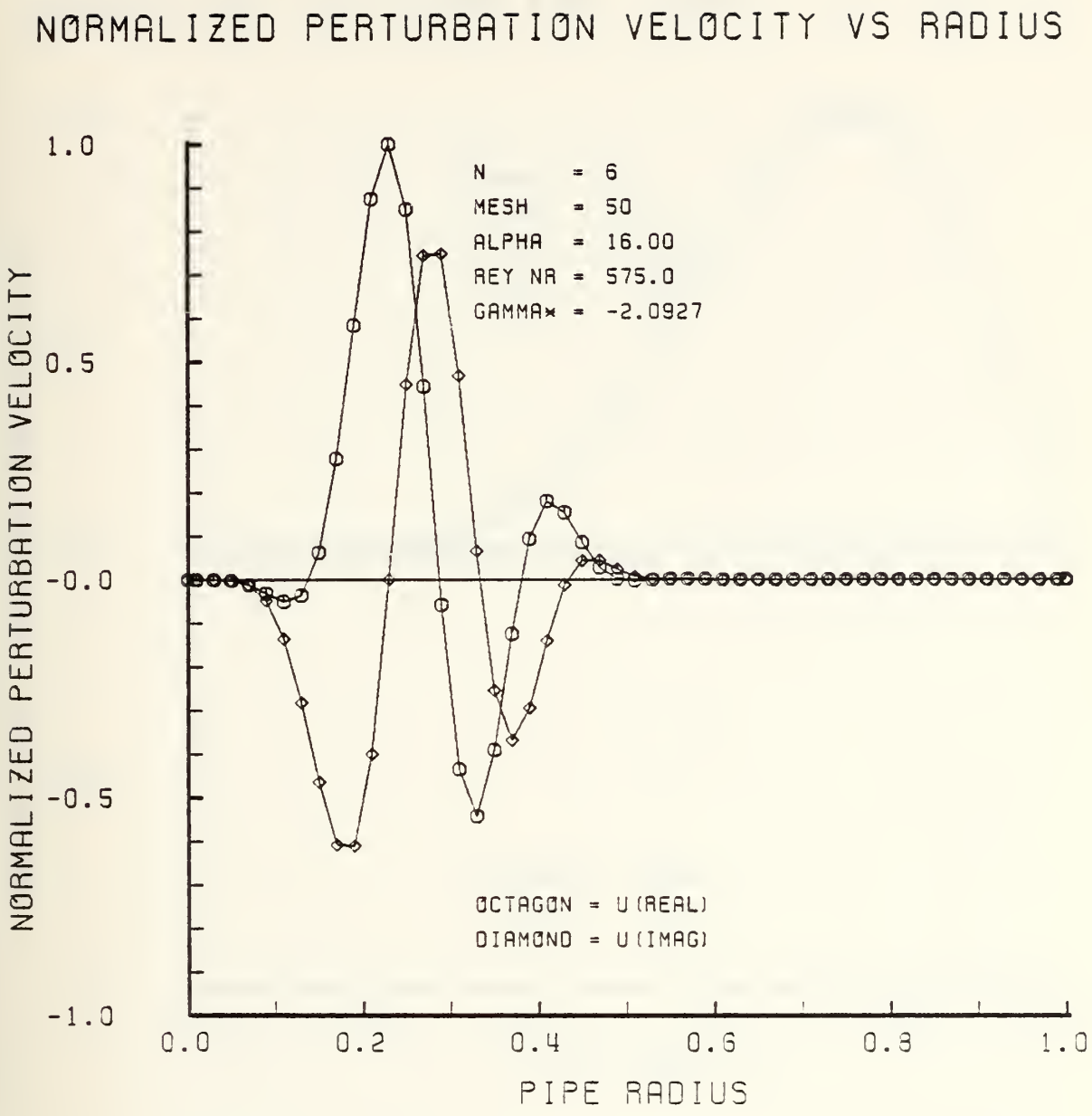
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 16.00
REY NR = 575.0
GAMMA× = -2.0927
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-29

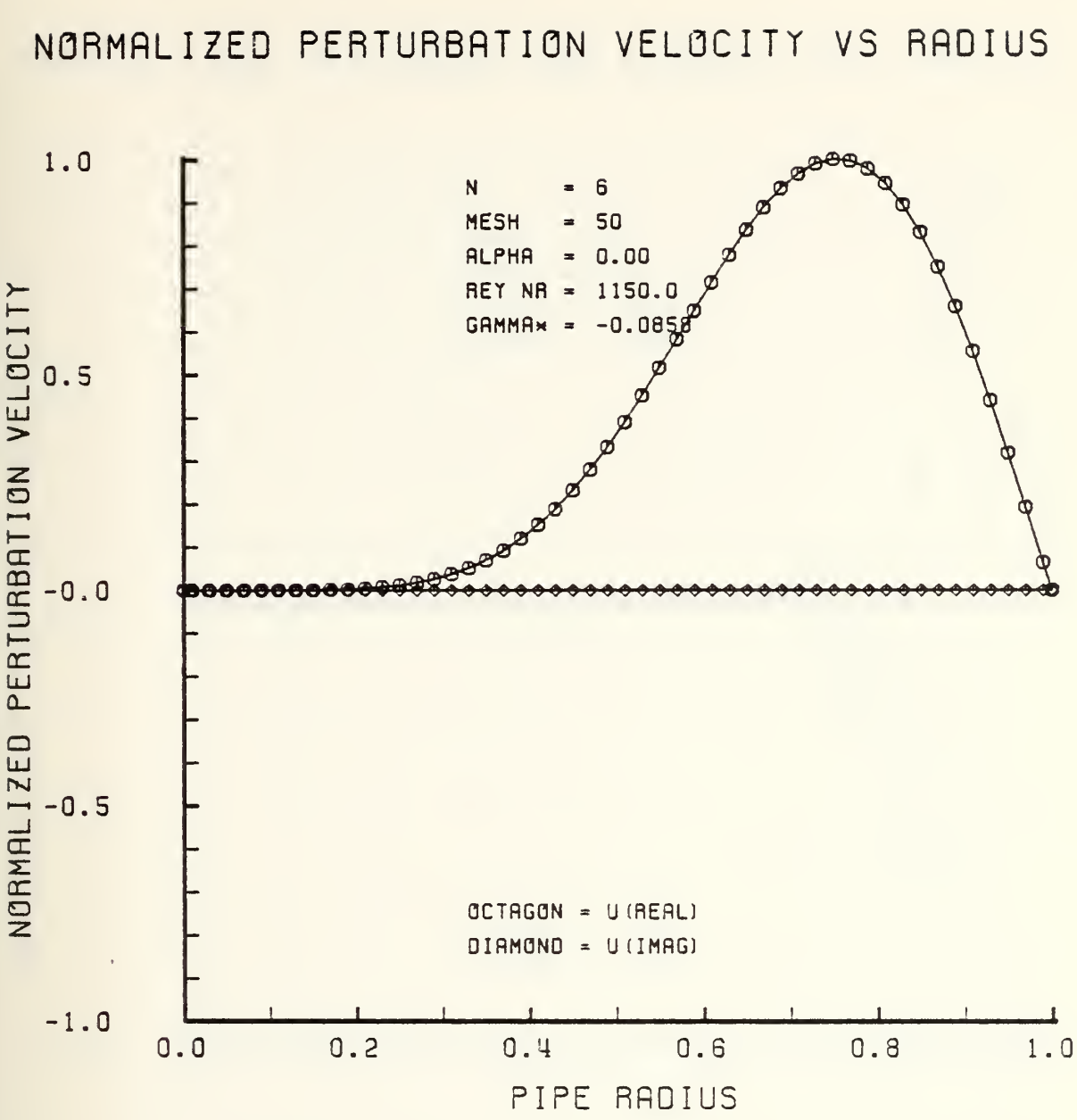
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 0.00
REY NR = 1150.0
GAMMA* = -0.0858
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-30

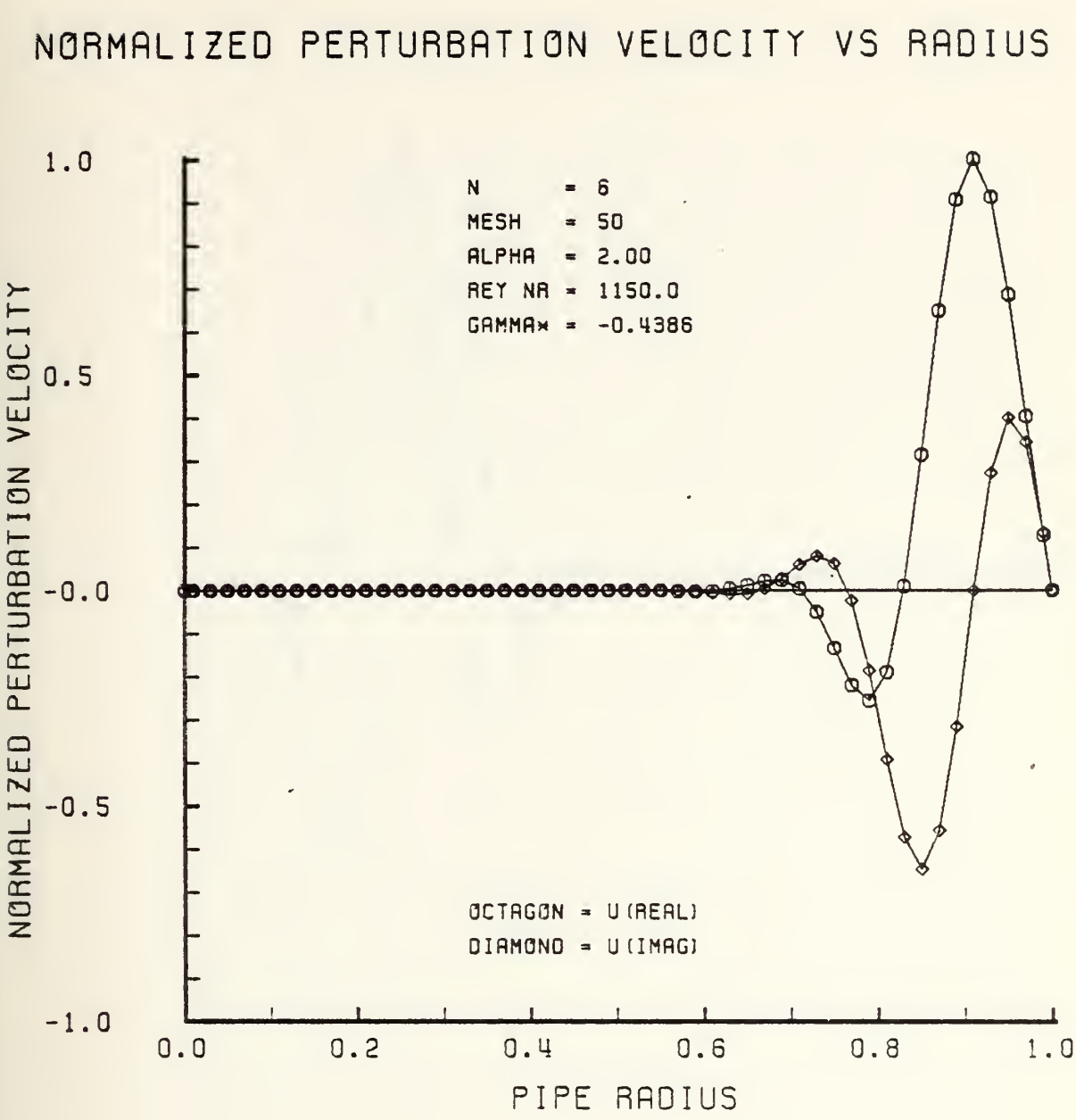
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 2.00
REY NR = 1150.0
GAMMA* = -0.4386
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-31

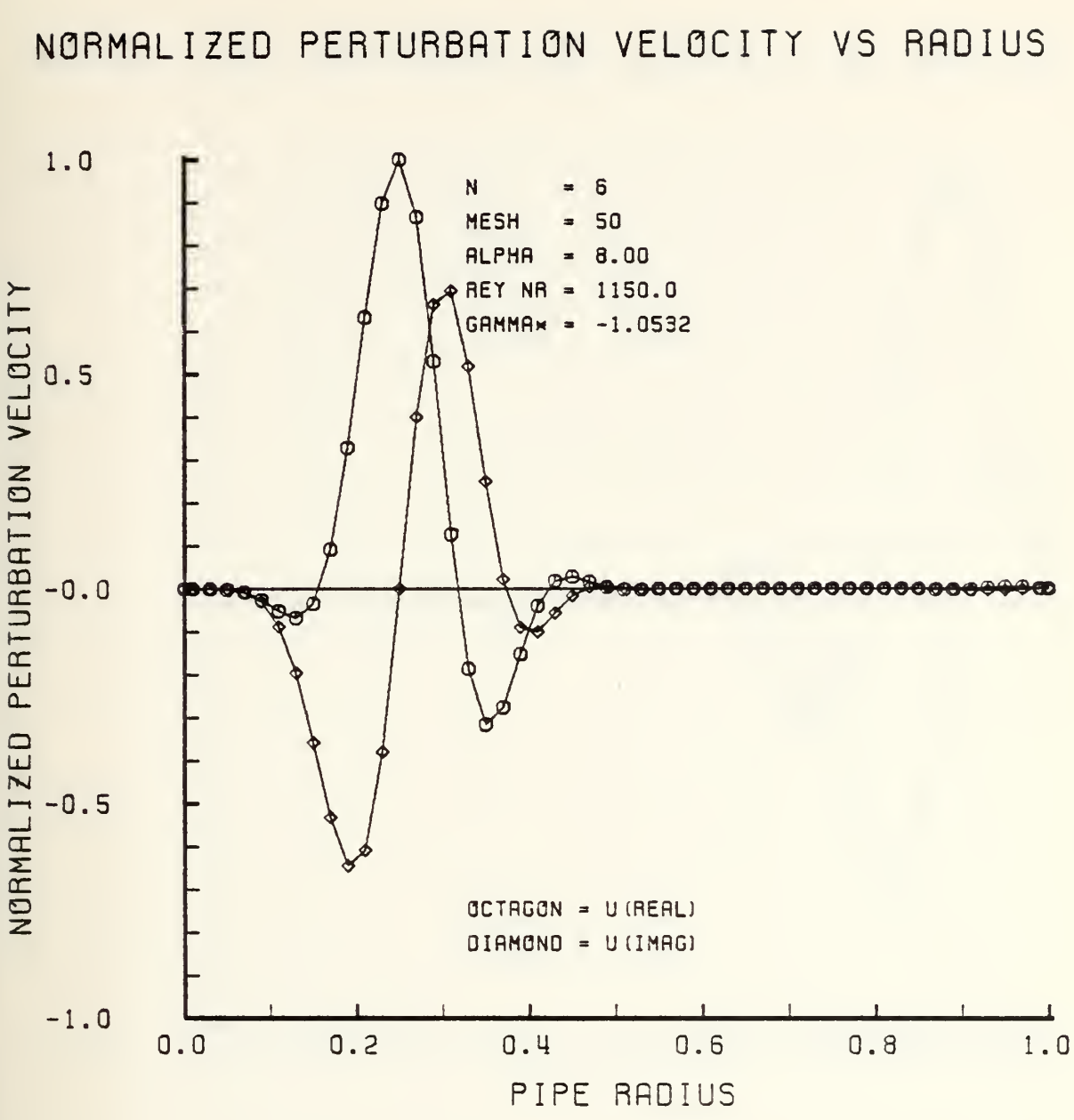
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 8.00
REY NR = 1150.0
GAMMA* = -1.0532
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-32

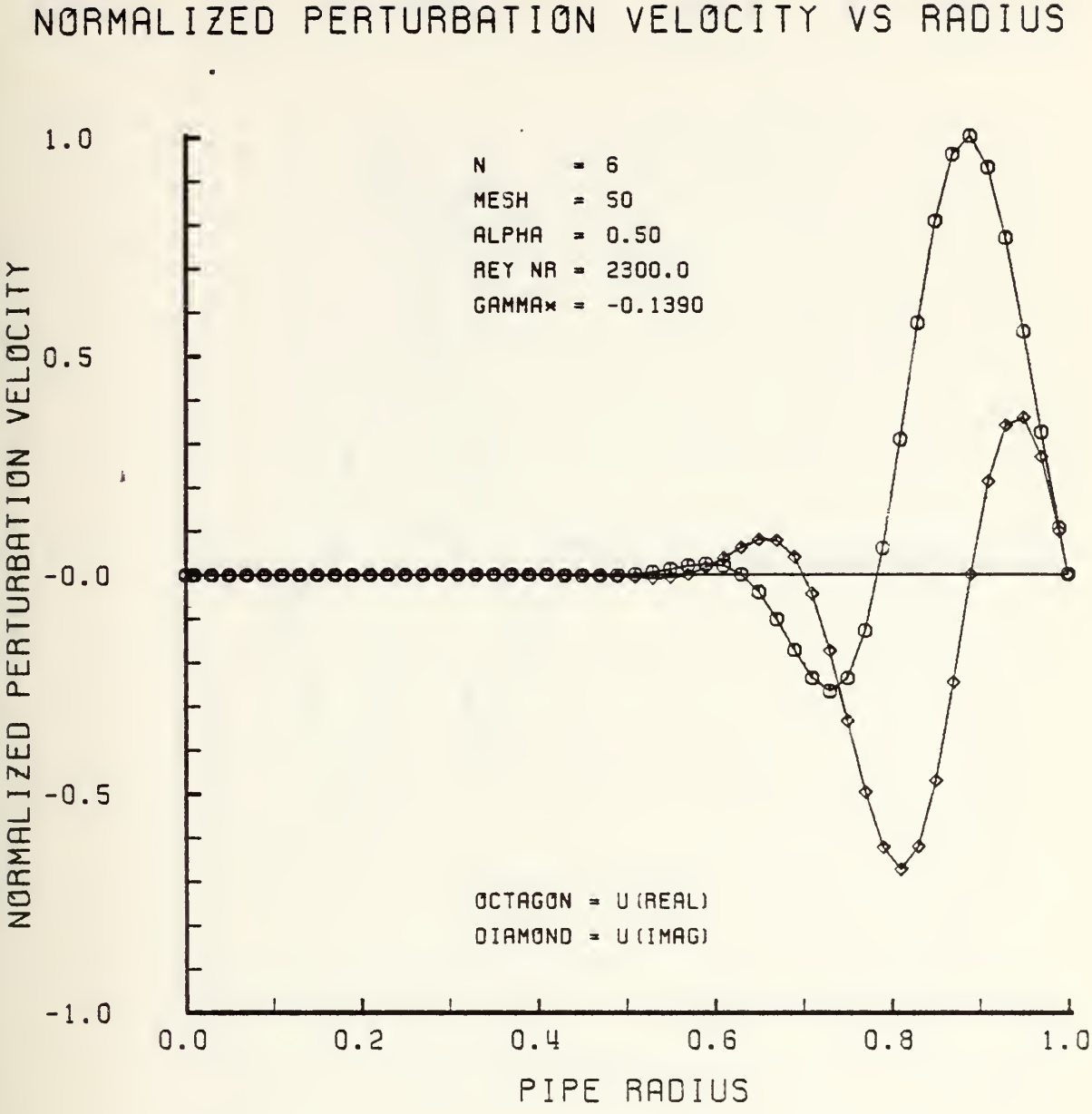
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 0.50
REY NR = 2300.0
GAMMA* = -0.1390
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-33

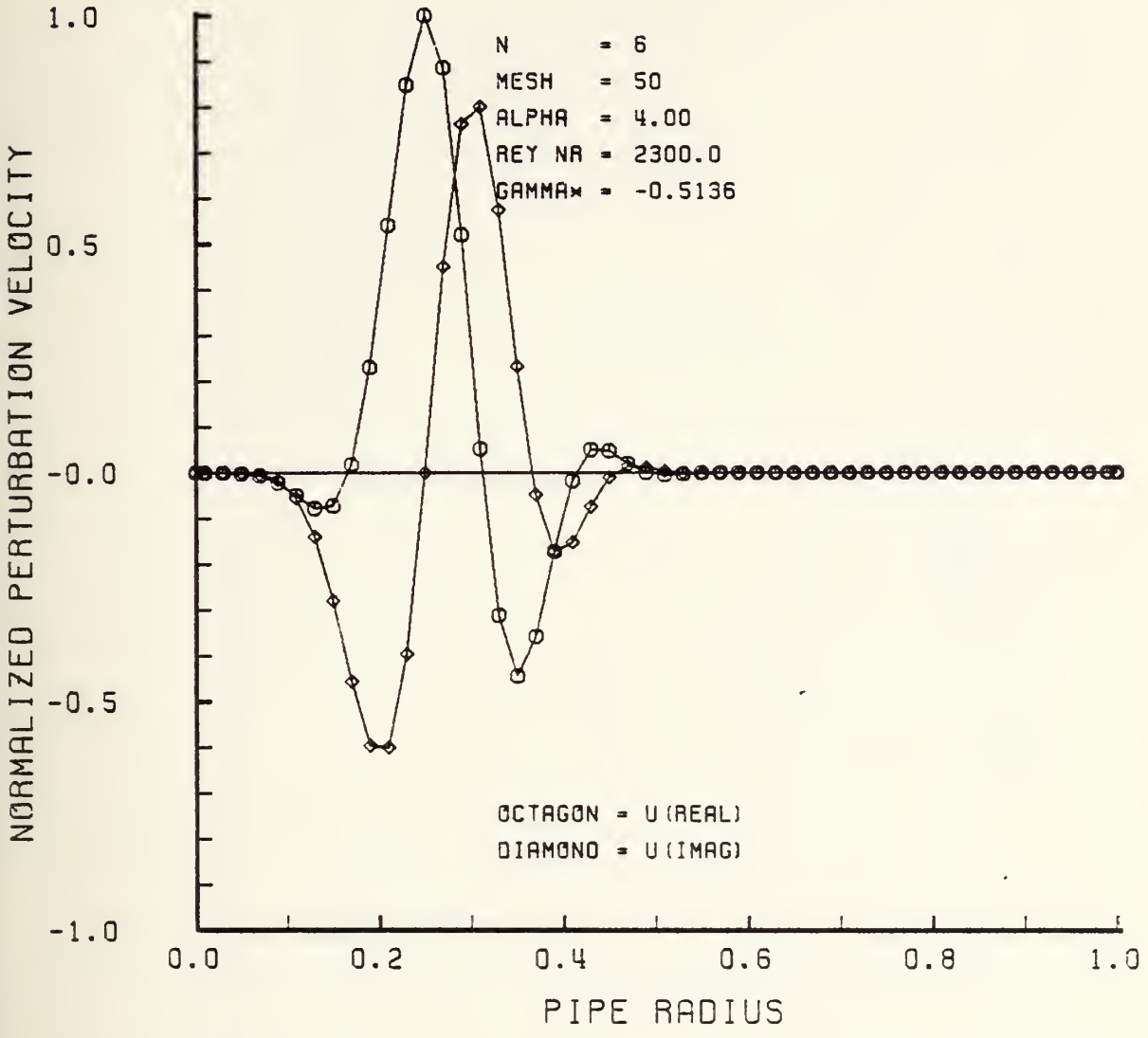
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 4.00
REY NR = 2300.0
GAMMAx = -0.5136
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-34

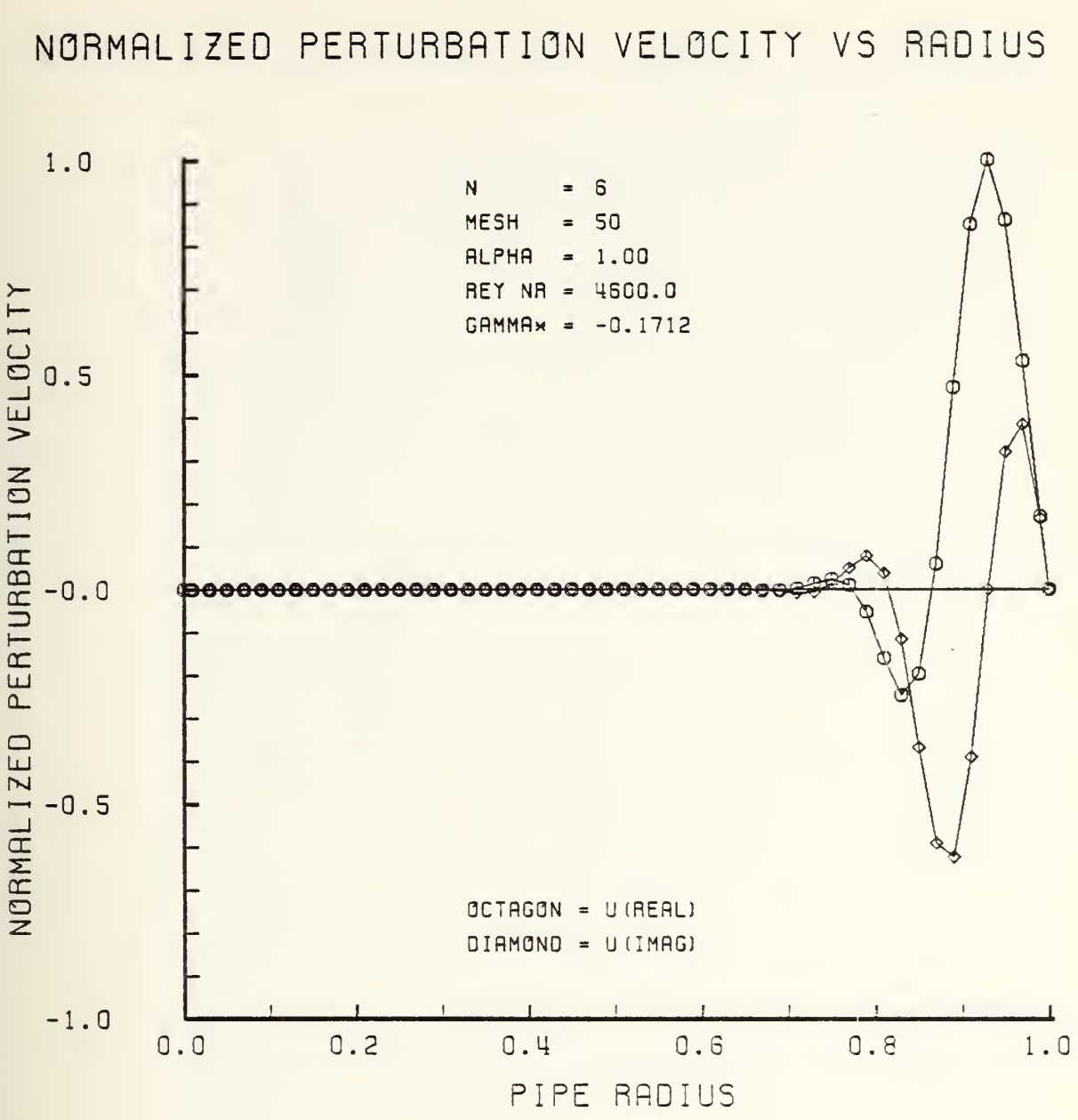
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 1.00
REY NR = 4600.0
GAMMAx = -0.1712
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-35

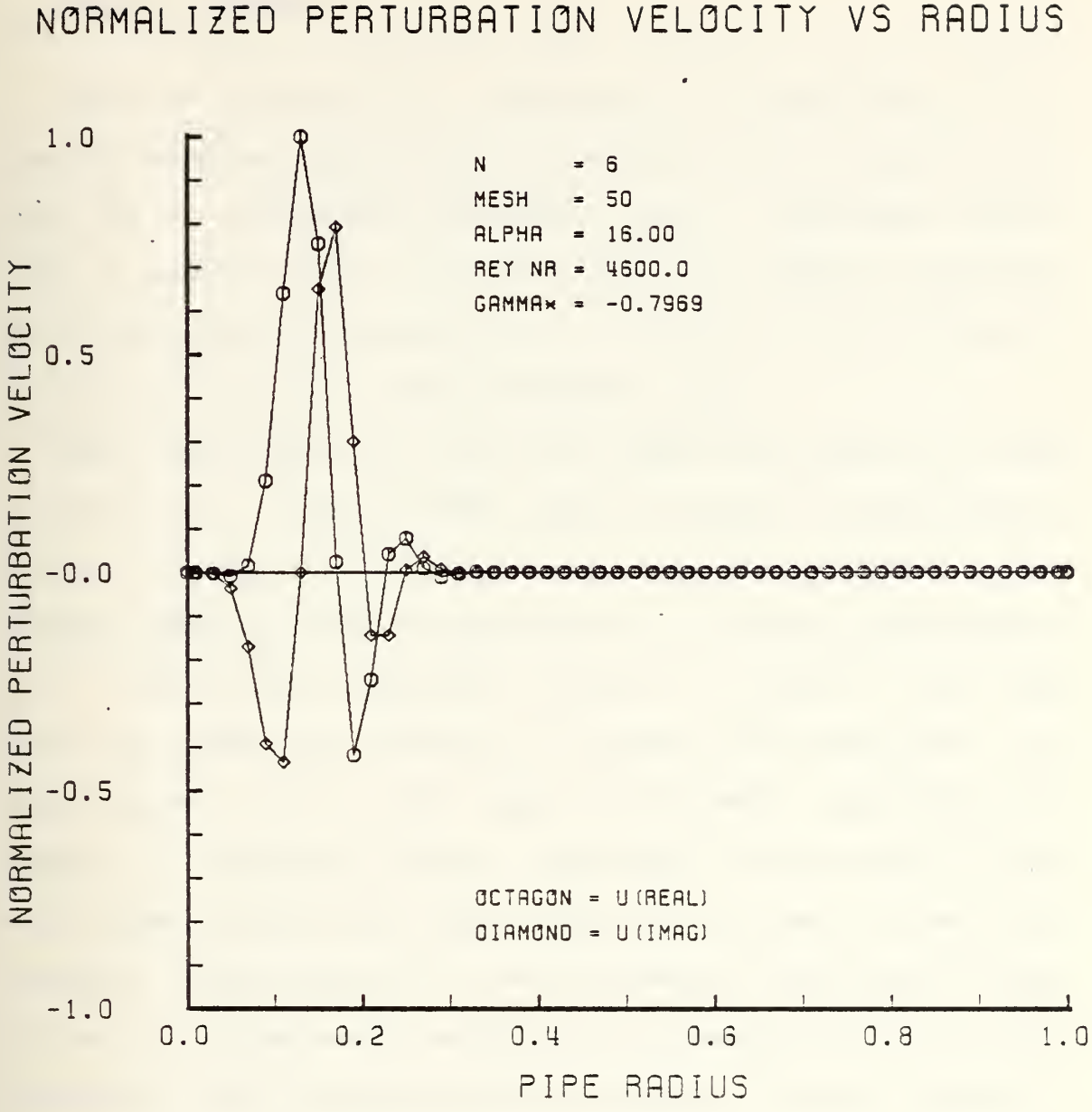
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 16.00
REY NR = 4600.0
GAMMA× = -0.7969
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-36

D. NUMERICAL ACCURACY

Numerical accuracy of the solutions of the governing vorticity transport equations was verified by substituting the least stable eigenvalue and its corresponding eigenvector into the eigensystem relation given in equation (5-39). The error vector was examined to determine the residual error of the solution. For the most part the error was of the order 10^{-12} to 10^{-9}, a most satisfactory result.

Upon closer examination of the error vector and the stability trends in the plots, it became evident that the accuracy of the numerical solution for the governing equations was not only dependent on the Reynolds number as previously mentioned but on the axial wave number as well. Plots of the perturbation velocity vs. radius for very large axial wave numbers are contained in Figures 6-37 through 6-39. The three figures, one for each angular wave number investigated, are examples of inaccurate solutions. Dependence of the solution on Reynolds number and axial wave number, became more evident as Re and α were increased. Rapid excursions of the perturbation velocity over a small portion of the pipe radius indicate that the computational mesh was not sufficiently fine to approximate the perturbation velocity function in the region of activity. This supposition can also be supported by examining the error vector where the residual error was found to be of the order 10^{-1} or worse, in the region of perturbation activity. Recall for $n = 1$ that some of the activity remained at the wall as α was increased. Refer to Figures 6-20 through 6-23. The residual error was

nominally of the order 10^{-5} in this region. Therefore, the true representation of the perturbation velocity is directly dependent upon an accurate numerical solution of the governing equations. These inaccuracies can be resolved by using a nonuniform mesh in the region of interest as suggested by Arnold. It is suspected, however, that after a thorough analysis of the effects of the problem parameters and mesh size on the solution have been investigated, only small values of the axial wave number below about 10.0 and Reynolds numbers in the vicinity of 1150 will be of primary interest. It is felt by this investigator that a mesh size of N = 50 and the wide range of problem parameters used here yielded solutions that satisfied the governing vorticity transport equations to a high degree and were sufficient to show stability trends.

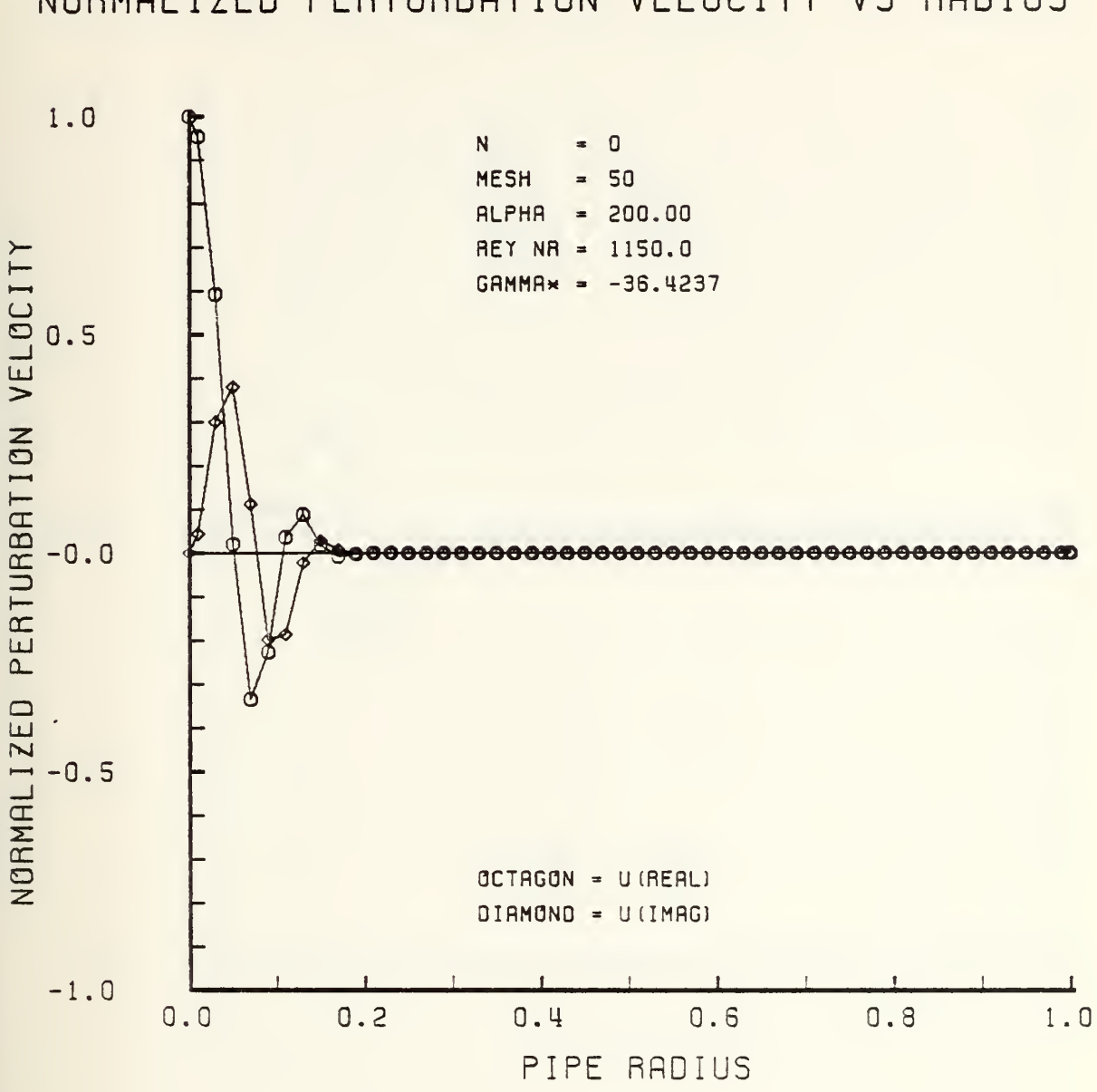
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 200.00
REY NR = 1150.0
GAMMA× = -36.4237
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-37

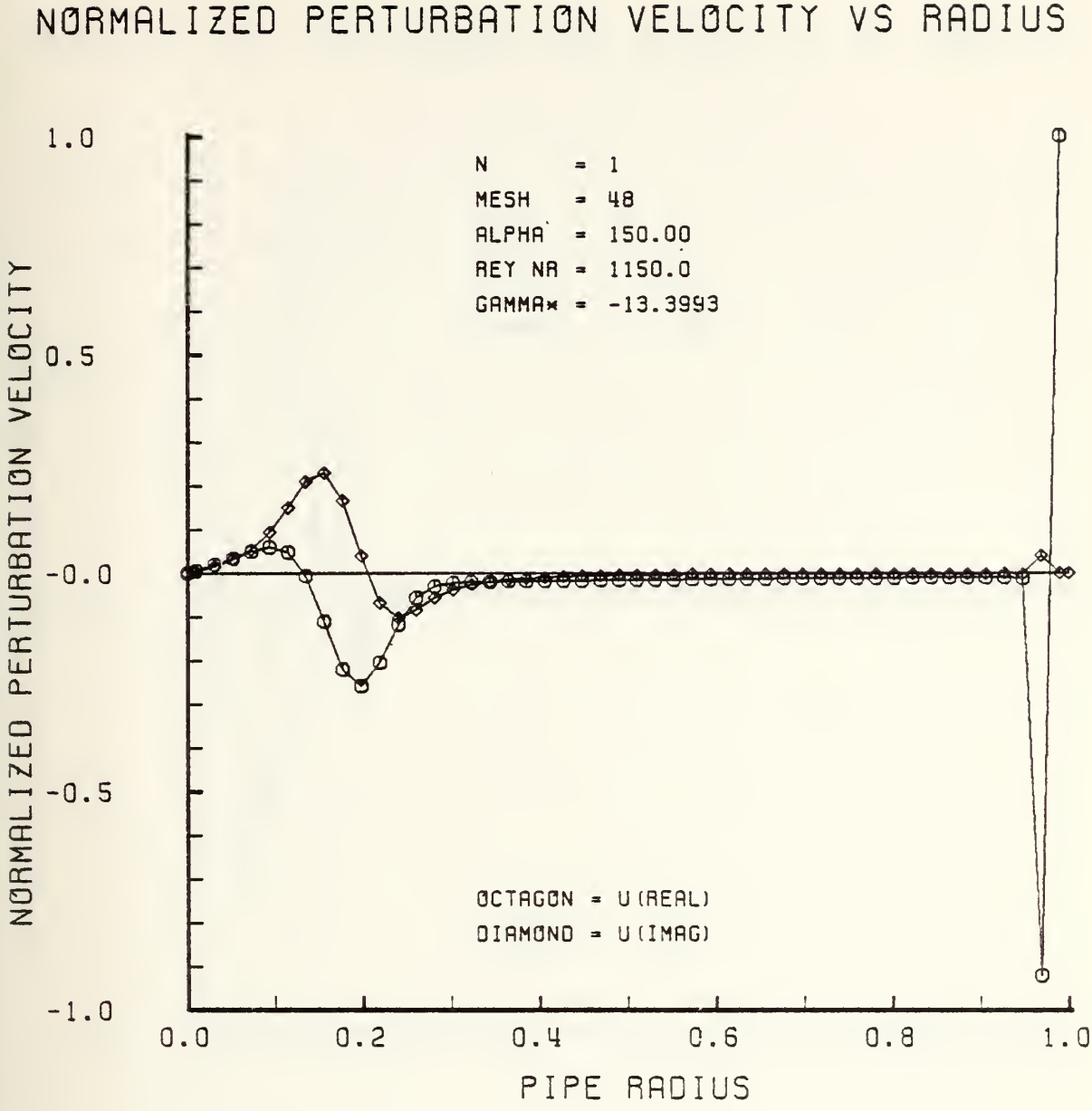
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 150.00
REY NR = 1150.0
GAMMA* = -13.3993
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-38

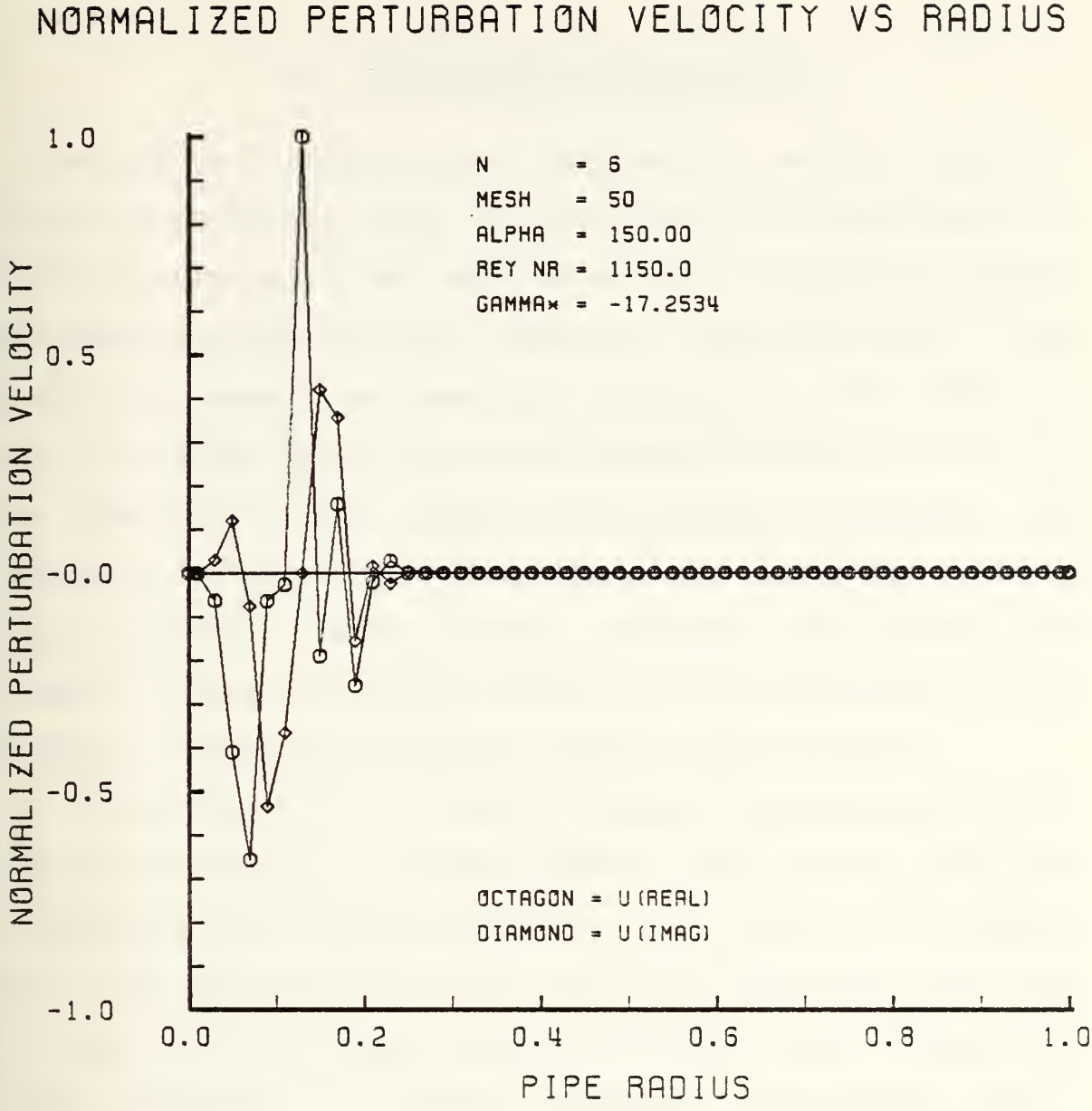
NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 150.00
REY NR = 1150.0
GAMMA× = -17.2534
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-39

VII. RECOMMENDATIONS AND CONCLUSIONS

The problem of predicting the transition from laminar to turbulent flow of incompressible fluids in circular pipes has been pursued for nearly 100 years and has been presented here in a straightforward manner. The governing equations for the dynamics of this problem consist of the Navier-Stokes equation and the continuity equation. After taking the curl of the Navier-Stokes equation and applying the continuity principle, the linearized vorticity transport vector equation is obtained. The velocity vector potential function, which contains functions of r and a complex exponential factor, was then introduced. The resulting two linearily independent equations form the basis for the solution of the problem in the form of the vorticity transport matrix equation.

A valid solution of the governing equations was made possible because of two factors; (1) the advancements in the linearized theory by introducing a purely imaginary exponential form of the axial wave number, and (2) the rigorous method in which the boundary conditions at the pipe axis were derived and those constraints enforced. After an appropriate change of variables was performed, the governing equations were transformed into an equivalent pair of coupled, homogeneous differential equations that were solved in eigenvalue problem format. Accurate solutions of the governing equations were achieved by the refinement and combination of several standard methods of numerical analysis. These methods include; (1) the approximation of the governing equations along

a computational radial mesh developed by the half-station method, (2) the use of central and noncentral finite difference approximations having consistent fourth order truncation error, and (3) the use of complex, double precision number representations.

It was this author's intention from the outset to make this work as complete as possible. This was accomplished by including the development of the theory, the derivation of the boundary conditions, and a detailed explaination of the numerical methods used as well as obtaining solutions to the governing equations. This will enable future investigators to use this thesis as a starting point and as a single source reference for follow-on studies of this subject.

The conclusions of this numerical analysis follow directly from the results of the pipe flow stability problem which were discussed thoroughly in the previous chapter.

The most significant findings are:

(1) for n = 0, the flow is stable at all Re and α.

(2) for n = 1, the flow appears to be unstable at all Re for small values of α. This result is unfortunately in direct conflict with experimental observations.

(3) for n = 1, unstable eigenvalues are associated with perturbation activity at the wall.

(4) for n = 6, the flow is stable at all Re and α.

The following observations are also of interest:

(5) for n = 0, 1, and 6, increasing Reynolds number has a destabilizing effect on the flow.

(6) for n = 0, 1, and 6, increasing axial wave number has a stabilizing effect on the flow for fixed Re.

(7) for n = 1 and 6, increasing axial wave number causes a shift in the perturbation activity from the wall to the axis for fixed Re.

The results for angular wave numbers n = 1 and 6 are believed to be new. Conclusions (1), (5), (6), and (7) are in agreement with experimental results or previous theoretical investigations.

It is this author's opinion that the numerical analysis of the vorticity transport equations is now complete for angular wave numbers n = 0 and 6. It is felt, however, that the numerical results for angular wave number n = 1 warrant further study. The unresolved paradox is quite puzzeling in that for n = 1 the flow appears to be unstable at all Reynolds numbers. Continued numerical investigations for n = 1 and follow-on studies of other angular wave numbers should prove fruitful in establishing a theory which predicts the transition from laminar to turbulent flow of fluids in pipes.

APPENDIX A

COEFFICIENTS OF THE VORTICITY TRANSPORT EQUATIONS

The generalized coefficient matrices which appear in the basic vorticity transport equations (2-40) are defined below.

$$[M_4] = \frac{1}{Re} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{A-1}$$

$$[M_3] = \frac{1}{Re} \begin{bmatrix} 0 & \frac{in}{r} \\ 0 & -\frac{2}{r} \end{bmatrix} \tag{A-2}$$

$$[M_2] = \frac{1}{Re} \begin{bmatrix} \left(\frac{n^2}{r^2} + \alpha^2\right) & \frac{2\,in}{r^2} \\ -\frac{4\,in}{r^2} & \left(\frac{2n^2+3}{r^2} + 2\alpha^2\right) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2i\alpha(1-r^2) \end{bmatrix} \tag{A-3}$$

$$[M_1] = \frac{1}{Re} \begin{bmatrix} -\left(\frac{n^2}{r^3} - \frac{\alpha^2}{r}\right) & -in\left(\frac{n^2+1}{r^3} + \frac{\alpha^2}{r}\right) \\ \frac{4\,in}{r^3} & -\left(\frac{2n^2+3}{r^3} - \frac{2\alpha^2}{r}\right) \end{bmatrix} + \begin{bmatrix} 0 & 2n\alpha\left(\frac{1}{r}-r\right) \\ 0 & 2i\alpha\left(\frac{1}{r}-r\right) \end{bmatrix} \tag{A-4}$$

$$[M_0] = \frac{1}{Re}\begin{bmatrix} -\left(\frac{n^2(n^2-1)}{r^4} + \frac{(2n^2+1)\,\alpha^2}{r^2} + \alpha^4\right) & -in\left(\frac{n^2-1}{r^4} + \frac{3\alpha^2}{r^2}\right) \\ 4\,in\left(\frac{n^2-1}{r^4} + \frac{\alpha^2}{r^2}\right) & -\left(\frac{(n^2+3)(n^2-1)}{r^4} + \frac{2(n^2+1)\,\alpha^2}{r^2} + \alpha^4\right) \end{bmatrix}$$

$$+ \begin{bmatrix} -2i\alpha\left(\frac{n^2}{r^2} - (n^2-\alpha^2) - \alpha^2 r^2\right) & 2n\alpha\left(\frac{1}{r^2} + 1\right) \\ -4n\alpha\left(\frac{1}{r^2} - 1\right) & -2i\alpha\left(\frac{n^2+1}{r^2} - (n^2+1-\alpha^2) - \alpha^2 r^2\right) \end{bmatrix} \qquad \text{(A-5)}$$

$$[N_2] = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{(A-6)}$$

$$[N_1] = \begin{bmatrix} 0 & \frac{in}{r} \\ 0 & -\frac{1}{r} \end{bmatrix} \qquad \text{(A-7)}$$

$$[N_0] = \begin{bmatrix} \left(\frac{n^2}{r^2} + \alpha^2\right) & \frac{in}{r^2} \\ -\frac{2\,in}{r^2} & \left(\frac{n^2+1}{r^2} + \alpha^2\right) \end{bmatrix} \qquad \text{(A-8)}$$

APPENDIX B

COEFFICIENTS OF THE TRANSFORMED VORTICITY TRANSPORT EQUATIONS FOR n = 0, 1, and 6

A. COEFFICIENTS FOR n = 0

Since the vorticity transport equations uncouple for n = 0, only the solution for the eigenfunction Q(r) is sought. Therefore the details of the complete coefficient matrices are not required. The coefficients which appear here represent the matrix element (2,2) and correspond to the primed quantities of equation (2-43).

$$M_4' = - \frac{r}{Re} \tag{B-1}$$

$$M_3' = - \frac{6}{Re} \tag{B-2}$$

$$M_2' = \frac{1}{Re} \left(- \frac{3}{r} + 2\alpha^2 r \right) + 2i\alpha \, (r - r^3) \tag{B-3}$$

$$M_1' = \frac{1}{Re} \left(\frac{3}{r^2} + 6\alpha^2 \right) + 6i\alpha \, (1 - r^2) \tag{B-4}$$

$$M_0' = - \frac{\alpha^4 r}{Re} + 2i\alpha^3 \, (r^3 - r) \tag{B-5}$$

$$N_2' = - r \tag{B-6}$$

$$N_1' = - 3 \tag{B-7}$$

$$N_0' = \alpha^2 r \tag{B-8}$$

These coefficients also appear and are computed in part III of the main investigative computer program for n = 0.

B. COEFFICIENTS for n = 1

Since the vorticity transport equations do not uncouple for n = 1, the eigenfunctions P(r) and Q(r) must be solved for simultaneously. With the introduction of the additional parameter as a result of the change of variables, H(0) appears explicitly as an unknown in the system of equations. This requires the addition of two coefficient matrices to equation (2-40). The 2 X 1 column matrix $[M_5']$ appears on the left side of equation (2-47) and one 2 X 1 column matrix $[N_3']$ appears on the right side of equation (2-47). The complete coefficient matrices for n = 1 are defined below.

$$[M_4'] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{r^2}{Re} \end{bmatrix} \tag{B-9}$$

$$[M_3'] = \begin{bmatrix} 0 & \frac{ir}{Re} \\ 0 & -\frac{10r}{Re} \end{bmatrix} \tag{B-10}$$

$$[M_2'] = \begin{bmatrix} \frac{1}{Re}(1 + \alpha^2 r^2) & \frac{8i}{Re} \\ -\frac{4i}{Re} & \frac{1}{Re}(2\alpha^2 r^2 - 19) + 2i\alpha(r^2 - r^4) \end{bmatrix} \tag{B-11}$$

$$[M_1'] = \begin{bmatrix} \frac{1}{Re}\left(\frac{3}{r} + 5\alpha^2 r\right) & 2\alpha(r - r^3) \\ 0 & \frac{1}{Re}\left(\frac{3}{r} + 10\alpha^2 r\right) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{i}{Re}\left(\frac{12}{r} - \alpha^2 r\right) \\ -\frac{12i}{Rer} & 10i\alpha\,(r - r^3) \end{bmatrix} \qquad \text{(B-12)}$$

$$[M_0'] = \begin{bmatrix} \frac{\alpha^2}{Re}\,(1-\alpha^2 r^2) & 2\alpha(3 - r^2) \\ -\,4\alpha(1 - r^2) & \frac{1}{Re}\,(4\alpha^2 - \alpha^4 r^2) \end{bmatrix}$$

$$+ \begin{bmatrix} -2i\alpha(1 - r^2 + \alpha^2 r^2 - \alpha^2 r^4) & \frac{-5i\alpha^2}{Re} \\ \frac{4i\alpha^2}{Re} & 2i\alpha\{2(1 - r^2) - \alpha^2(r^2 - r^4)\} \end{bmatrix} \qquad \text{(B-13)}$$

$$[M_5'] = \begin{bmatrix} 2\alpha(2 - \alpha^2 + \alpha^2 r^2) + \frac{i\alpha^4}{Re} \\ -\,\frac{\alpha^4}{Re} - 2i\alpha^3\,(1 - \alpha r^2) \end{bmatrix} \qquad \text{(B-14)}$$

$$[N_2'] = \begin{bmatrix} 0 & 0 \\ 0 & -r^2 \end{bmatrix} \tag{B-15}$$

$$[N_1'] = \begin{bmatrix} 0 & ir \\ 0 & -5r \end{bmatrix} \tag{B-16}$$

$$[N_0'] = \begin{bmatrix} 1 + \alpha^2 r^2 & 3i \\ -2i & -2 + \alpha^2 r^2 \end{bmatrix} \tag{B-17}$$

$$[N_3'] = \begin{bmatrix} -i\alpha^2 \\ \alpha^2 \end{bmatrix} \tag{B-18}$$

These coefficients also appear and are computed in Part III of the main investigative computer program for n = 1.

C. COEFFICIENTS for n = 6

The vorticity transport equations for n = 6 remain coupled. There are no additional parameters introduced with the change of variables to P(r) and Q(r). The system of equations take the form of equation (2-40) once again. The complete coefficient matrices for n = 6 are defined below.

$$[M_4'] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{r^3}{Re} \end{bmatrix} \tag{B-19}$$

$$[M_3'] = \begin{bmatrix} 0 & \frac{6ir^2}{Re} \\ 0 & -\frac{14r^2}{Re} \end{bmatrix} \tag{B-20}$$

$$[M_2'] = \begin{bmatrix} \frac{r^2}{Re}(36 + \alpha^2 r^2) & 0 \\ 0 & \frac{r}{Re}(21 + 2\alpha^2 r^2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{66ir}{Re} \\ -\frac{24ir^2}{Re} & 2i\alpha(r^3 - r^5) \end{bmatrix} \tag{B-21}$$

$$[M_1'] = \begin{bmatrix} \frac{1}{Re}(252r + 9\alpha^2r^2) & 12\alpha(r^2 - r^4) \\ 0 & \frac{1}{Re}(315 + 14\alpha^2r^2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\frac{i}{Re}(42 + 6\alpha^2r^2) \\ -\frac{168ir}{Re} & 14i\alpha(r^2 - r^4) \end{bmatrix} \qquad \text{(B-22)}$$

$$[M_0'] = \begin{bmatrix} -\frac{1}{Re}(972 + 57\alpha^2r^2 + \alpha^4r^4) & 24\alpha(2r - r^3) \\ -24\alpha(r^2 - r^4) & -\frac{1}{Re}\left(\frac{1152}{r} + 56\alpha^2r + \alpha^4r^3\right) \end{bmatrix}$$

$$+ \begin{bmatrix} -2i\alpha r^2\{36-36r^2 + \alpha^2r^2(1-r^2)\} & -\frac{i}{Re}\left(\frac{768}{r} + 36\alpha^2r\right) \\ \frac{i}{Re}(648 + 24\alpha^2r^2) & -56i\alpha(r-r^3)-2i\alpha^3(r^3-r^5) \end{bmatrix} \qquad \text{(B-23)}$$

$$[N_2'] = \begin{bmatrix} 0 & 0 \\ 0 & -r^3 \end{bmatrix} \qquad \text{(B-24)}$$

$$[N_1'] = \begin{bmatrix} 0 & 6ir^2 \\ 0 & -7r^2 \end{bmatrix} \tag{B-25}$$

$$[N_0'] = \begin{bmatrix} 36r^2 + \alpha^2 r^4 & 24ir \\ -12ir^2 & 28r + \alpha^2 r^3 \end{bmatrix} \tag{B-26}$$

These coefficients also appear and are computed in part III of the main investigative computer program for n = 6.

APPENDIX C

COEFFICIENTS OF THE BOUNDARY EQUATIONS AT THE AXIS

The matrices which appear in the boundary equations at the axis, equations (3-4) through (3-8) are defined below.

$$[C_1] = \begin{bmatrix} -n^2 & -in \\ +4in & -(n^2+3) \end{bmatrix} \qquad \text{(C-1)}$$

$$[C_2] = \begin{bmatrix} -n^2 & -2in \\ +4in & -(n^2+4) \end{bmatrix} \qquad \text{(C-2)}$$

$$[C_3] = \begin{bmatrix} -\frac{n^2}{2} & -\frac{3}{2}in \\ +2in & -\frac{1}{2}(n^2+3) \end{bmatrix} \qquad \text{(C-3)}$$

$$[C_4] = \begin{bmatrix} -\left\{2n^2 - (2n^2+1)\frac{i\alpha}{Re}\right\} & -in\left(2 - \frac{3i\alpha}{Re}\right) \\ +4in\left(1 - \frac{i\alpha}{Re}\right) & -2(n^2+1)\left(1 - \frac{i\alpha}{Re}\right) \end{bmatrix} \qquad \text{(C-4)}$$

$$[D_4] = \begin{bmatrix} + n^2 & + in \\ - 2\,in & + (n^2 + 1) \end{bmatrix} \tag{C-5}$$

$$[C_5] = \begin{bmatrix} - \frac{n}{6} & - \frac{2i}{3} \\ + \frac{2i}{3} & - \frac{n}{6} \end{bmatrix} \tag{C-6}$$

$$[C_6] = \begin{bmatrix} - n & - 2i \\ + 2i & - n \end{bmatrix} \tag{C-7}$$

$$[C_7] = \begin{bmatrix} - \frac{n^2}{24} & - \frac{5\,in}{24} \\ + \frac{in}{6} & - \frac{(n^2 - 5)}{24} \end{bmatrix} \tag{C-8}$$

$$[C_8] = \begin{bmatrix} - \left\{ n^2 - \left(n^2 - \frac{3}{2}\right) \frac{i\alpha}{Re} \right\} & - in \left(3 - \frac{5i\alpha}{2Re}\right) \\ + 2\,in \left(1 - \frac{i\alpha}{Re}\right) & - (n^2 - 3) \left(1 - \frac{i\alpha}{Re}\right) \end{bmatrix} \tag{C-9}$$

$$[D_8] = \begin{bmatrix} + \frac{n^2}{2} & + \frac{3\,in}{2} \\ - in & + \frac{(n^2 - 3)}{2} \end{bmatrix} \tag{C-10}$$

$$[C_9] = \begin{bmatrix} \left(\frac{i\alpha^3}{Re} + 2n^2 - 2\alpha^2\right) & -2in \\ -4in & \left(\frac{i\alpha^3}{Re} + 2n^2 + 2 - 2\alpha^2\right) \end{bmatrix} \quad \text{(C-11)}$$

$$[D_9] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(C-12)}$$

Column matrices used in equation (5-36) for n = 1.

$$[C_9^*] = \begin{bmatrix} \left(\frac{\alpha^3}{Re} - 4i + 2i\alpha^2\right) \\ \left(-2\alpha^2 + \frac{i\alpha^3}{Re}\right) \end{bmatrix} \{H(0)\} \quad \text{(C-13)}$$

$$[D_9^*] = \begin{bmatrix} -i \\ 1 \end{bmatrix} \{H(0)\} \quad \text{(C-14)}$$

APPENDIX D

SPECIAL CONDITIONS AT THE AXIS

The determinates are formed from the corresponding matrices as defined in Appendix C. For angular wave numbers $n \geqq 6$, all of the determinates become non-singular.

$$|C_1| = n^2 (n^2 - 1) \tag{D-1}$$

$$|C_2| = n^2 (n^2 - 4) \tag{D-2}$$

$$|C_3| = \frac{n^2}{4} (n^2 - 9) \tag{D-3}$$

$$|i\alpha[C_4] - \gamma[D_4]| = (n^2 - 1) \left\{ -2\alpha^2 \left(1 - \frac{i\alpha}{Re}\right) \left[2n^2 - (2n^2 - 1) \frac{i\alpha}{Re}\right] + \gamma i\alpha \left[4n^2 - (4n^2 - 1) \frac{i\alpha}{Re}\right] + \gamma^2 n^2 \right\} \tag{D-4}$$

$$|C_5| = \frac{1}{36} (n^2 - 16) \tag{D-5}$$

$$|C_6| = (n^2 - 4) \tag{D-6}$$

$$|C_7| = \frac{n^2}{576} (n^2 - 25) \tag{D-7}$$

$$|i\alpha[C_8] - \gamma[D_8]| = (n^2 - 9)\left\{-\alpha^2\left(1 - \frac{i\alpha}{Re}\right)\left[-\left(n^2 - \frac{1}{2}\right)\frac{i\alpha}{Re} + n^2\right]\right.$$

$$\left. + \frac{\gamma i\alpha^3}{2}\left[-\left(n^2 - \frac{1}{2}\right)\frac{i\alpha}{Re} + 2n^2\right] + \frac{\gamma^2\alpha^4 n^2}{4}\right\} \qquad (D\text{-}8)$$

```
//WALLACE JOB (2027,0084),'THESIS N = 0',CLASS=B                    PIP00010
//*MAIN LINES=(10)                                                  PIP00020
//  EXEC FRTXCLGP,IMSL=DP,REGION.GO=1024K                           PIP00030
//FORT.SYSIN DD *                                                   PIP00040
C                                                                   PIP00050
C*******************************************************************PIP00060
C                                                                   PIP00070
C       THIS PROGRAM WAS DEVELOPED TO PERFORM A NUMERICAL ANALYSIS  PIP00080
C       OF THREE DIMENSIONAL PIPE FLOW STABILITY FOR AXIAL WAVE     PIP00090
C       NUMBER N = 0. THE PROGRAM IS DIVIDED INTO EIGHT PARTS.      PIP00100
C       THE FIRST PART COMPUTES THE CENTRAL FINITE DIFFERENCE       PIP00110
C       COEFFICIENTS FOR THE DERIVATIVES OF Q(R) ONLY, AT THE       PIP00120
C       INTERIOR MESH POINTS ALONG THE PIPE RADIUS. IN THE SECOND   PIP00130
C       PART,  PRE-COMPUTED NON-CENTRAL FINITE DIFFERENCE           PIP00140
C       COEFFICIENTS FOR THE DERIVATIVES OF Q(R) AT POINTS          PIP00150
C       NEAR THE AXIS AND WALL ARE READ IN AS DATA. THE THIRD PART  PIP00160
C       COMPUTES THE VORTICITY TRANSPORT EQUATION COEFFICIENTS OF   PIP00170
C       Q(R) AND THEIR RESPECTIVE DERIVATIVES AT ALL  RADIAL MESH   PIP00180
C       POINTS (STATIONS). THE FOURTH PART COMPUTES THE A AND B     PIP00190
C       MATRIX ELEMENTS,  WHICH MAKE-UP THE EIGENSYSTEM TO BE SOLVED PIP00200
C       IN THE FORMAT,  A * X = GAMMA * B * X.  IN THE FIFTH PART   PIP00210
C       THE A AND B MATRICES COMPRISING THE EIGENSYSTEM ARE SOLVED  PIP00220
C       IN THE SUBROUTINE EIGZC.  THE RESULTING EIGENVALUES AND     PIP00230
C       CORRESPONDING EIGENVECTOR REPRESENT GAMMA AND THE COLUMN    PIP00240
C       VECTOR X,  RESPECTIVELY, AND ARE SOLUTIONS OF THE GOVERNING PIP00250
C       PARTIAL DIFFERENTIAL EQUATIONS.  THE SIXTH PART VERIFIES    PIP00260
C       THE SOLUTION OF THE PDE AND THE MAGNITUDE OF THE RESIDUAL   PIP00270
C       IS DETERMINED.  THE SEVENTH PART COMPUTES THE NORMALIZED    PIP00280
```

```
C
C       AXIAL PERTURBATION VELOCITY U(R).  THE EIGHTH AND LAST
C       PART PLOTS THE PERTURBATION VELOCITY VERSUS PIPE RADIUS.
C       REQUIRED SUBROUTINES ARE INCLUDED AT THE END OF THE PROGRAM
C       LISTING.  THIS IS A GENERALIZED PROGRAM FOR VARIABLE SIZED
C       MESH UP TO A MAXIMUM MESH SIZE OF 50.  THIS PROGRAM MAY BE
C       MODIFIED TO RUN IN THE VM/CMS (TIME SHARING) MODE AS WELL
C       AS THE BATCH MODE (MVS).  WITH APPROPRIATE MODIFICATIONS
C       TO THE PROGRAM DIMENSION STATEMENTS, A MAXIMUM MESH SIZE
C       OF 100 MAY BE USED.  AVERAGE CPU TIME REQUIRED FOR ONE SET
C       OF DATA,  I.E. ONE AXIAL WAVE NUMBER AND ONE REYNOLDS NUMBER
C       IS 8.1 SECONDS.
C*************************************************************
C
C       DEFINE PROGRAM VARIABLES AND ARRAYS
C
      COMPLEX*16 ZERO,ONE,AAXIS(4,3,6),AWALL(4,3,6),A1DER(7),A2DER(7)
      COMPLEX*16 A3DER(7),A4DER(7),A5DER,A(50,50),B(50,50)
      COMPLEX*16 D4(50),D3(50),D2(50),D1(50),D0(50),E2(50),E1(50),E0(50)
      COMPLEX*16 D3TEMP,E1TEMP,RAD(52)
      COMPLEX*16 EIGA(50),EIGB(50),EIGVEC(50,50),WK(1,1),EIGVAL(50)
      COMPLEX*16 TWO,AA(50,50),BB(50,50),Q(50),DQ(50),U(52)
      COMPLEX*16 Q06,A6CPX,UNORM(52),ERROR(100),SUMA,SUMB
      REAL*8 UAMP(52),AMPMAX,AMTEMP,UAMNOR(52),A6INV(6)
      REAL*8 URDP(52),UIDP(52)
      REAL*8 GAMR(50),GAMMAX,DMESH
      REAL*8 H4TH,H3RD,H2ND,H1ST,HTEMP,ATEMP,AATEMP,WATEMP,R,RE,ALPHA
      REAL*8 SIGN,R,R2,R3,RE,ALPHA
      REAL*8 DTEMPR,DTEMPI,ETEMP,AL22,AL2,AL62,AL6,AL23,ALSQ,AL3RD,AL4TH
      REAL*4 RADR(52),UR(52),UI(52),AMPNOR(52)
      REAL*4 X1(2),Y1(2),ALPLOT,REPLOT,GAMPLT,RMESH
      INTEGER MESH,M,MM,M2,IMAX
C
C       INITIALIZE THE COEFFICIENT ARRAYS AND SEVERAL CONSTANTS
C
      DATA ZERO/(0.0D0,0.0D0)/,ONE/(1.0D0,0.0D0)/,TWO/(2.0D0,0.0D0)/
      DATA AAXIS/72*(0.0D0,0.0D0)/,AWALL/72*(0.0D0,0.0D0)/
```

```
      DATA A1DER/7*(0.0D0,0.0D0)/,A2DER/7*(0.0D0,0.0D0)/                PIP00970
      DATA A3DER/7*(0.0D0,0.0D0)/,A4DER/7*(0.0D0,0.0D0)/                PIP00980
      DATA X1(1)/0.0/,Y1(1)/0.0/,X1(2)/1.0/,Y1(2)/0.0/                  PIP00990
      DATA MESH/50/                                                     PIP01000
C                                                                       PIP01010
C*************************************************************        PIP01020
C                                                                       PIP01030
C                              PART I                                   PIP01040
C                                                                       PIP01050
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THE                PIP01060
C                                                                       PIP01070
C        DERIVATIVES OF Q(R) AT INTERIOR MESH POINTS.                   PIP01080
C                                                                       PIP01090
C                                                                       PIP01100
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FOURTH ORDER       PIP01110
C                                                                       PIP01120
C        DERIVATIVE OF Q(R)  AT POINTS  4 - 47                          PIP01130
C                                                                       PIP01140
      DMESH = DFLOAT(MESH)                                              PIP01150
      H4TH = DMESH ** 4                                                 PIP01160
      ATEMP = (-1.0D0/6.0D0) * H4TH                                     PIP01170
      A4DER(1) = DCMPLX(ATEMP,0.0D0)                                    PIP01180
      A4DER(7) = A4DER(1)                                               PIP01190
      ATEMP = 2.0D0 * H4TH                                              PIP01200
      A4DER(2) = DCMPLX(ATEMP,0.0D0)                                    PIP01210
      A4DER(6) = A4DER(2)                                               PIP01220
      ATEMP = -6.5D0 * H4TH                                             PIP01230
      A4DER(3) = DCMPLX(ATEMP,0.0D0)                                    PIP01240
      A4DER(5) = A4DER(3)                                               PIP01250
      ATEMP = (56.0D0/6.0D0) * H4TH                                     PIP01260
      A4DER(4) = DCMPLX(ATEMP,0.0D0)                                    PIP01270
C                                                                       PIP01280
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THIRD ORDER        PIP01290
C                                                                       PIP01300
C        DERIVATIVE AT POINTS  4 - 47                                   PIP01310
C                                                                       PIP01320
      H3RD = DMESH ** 3                                                 PIP01330
      ATEMP = 0.125D0 * H3RD                                            PIP01340
      A3DER(1) = DCMPLX(ATEMP,0.0D0)                                    PIP01350
      A3DER(7) = -A3DER(1)                                              PIP01360
      A3DER(6) = DCMPLX(H3RD,0.0D0)                                     PIP01370
      A3DER(2) = -A3DER(6)                                              PIP01380
      ATEMP = (13.0D0/8.0D0) * H3RD                                     PIP01390
      A3DER(3) = DCMPLX(ATEMP,0.0D0)                                    PIP01400
      A3DER(5) = -A3DER(3)                                              PIP01410
C                                                                       PIP01420
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR SECOND ORDER       PIP01430
C                                                                       PIP01440
```

```
C         DERIVATIVE AT POINTS  2 - 48                                    PIP01450
C                                                                         PIP01460
      H2ND = DMESH ** 2                                                   PIP01470
      ATEMP = (-1.0D0/12.0D0) * H2ND                                      PIP01480
      A2DER(2) = DCMPLX(ATEMP,0.0D0)                                      PIP01490
      A2DER(6) = A2DER(2)                                                 PIP01500
      ATEMP = (16.0D0/12.0D0) * H2ND                                      PIP01510
      A2DER(3) = DCMPLX(ATEMP,0.0D0)                                      PIP01520
      A2DER(5) = A2DER(3)                                                 PIP01530
      ATEMP = -2.5D0 * H2ND                                               PIP01540
      A2DER(4) = DCMPLX(ATEMP,0.0D0)                                      PIP01550
C                                                                         PIP01560
CCC       COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FIRST ORDER         PIP01570
CCC                                                                       PIP01580
CCC       DERIVATIVE AT POINTS  2 - 48                                    PIP01590
C                                                                         PIP01600
      H1ST = DMESH                                                        PIP01610
      ATEMP = (1.0D0/12.0D0) * H1ST                                       PIP01620
      A1DER(2) = DCMPLX(ATEMP,0.0D0)                                      PIP01630
      A1DER(6) = -A1DER(2)                                                PIP01640
      ATEMP = (-8.0D0/12.0D0) * H1ST                                      PIP01650
      A1DER(3) = DCMPLX(ATEMP,0.0D0)                                      PIP01660
      A1DER(5) = -A1DER(3)                                                PIP01670
C                                                                         PIP01680
CC*************************************************************          PIP01690
C                                                                         PIP01700
CCC                          PART II                                      PIP01710
CCC                                                                       PIP01720
CCC       READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR    PIP01730
CCC                                                                       PIP01740
CCC       ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE AXIS.                PIP01750
CCC                                                                       PIP01760
CCC       BOUNDARY CONDITIONS AT THE AXIS ARE: D3Q(0)=0 AND DQ(0)=0.      PIP01770
CCC                                                                       PIP01780
CCC       FOR SECOND EQUATION IN Q(R) ONLY, FOR N = 0.                    PIP01790
C                                                                         PIP01800
   51 READ(5,59) I,J,K                                                    PIP01810
      IF(I.EQ.9) GO TO 54                                                 PIP01820
      HTEMP = DMESH ** I                                                  PIP01830
      DO 52 KKK = 1,K                                                     PIP01840
      READ(5,* ) AATEMP                                                   PIP01850
      AATEMP = AATEMP * HTEMP                                             PIP01860
      AAXIS(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                               PIP01870
   52 CONTINUE                                                            PIP01880
      GO TO 51                                                            PIP01890
   54 CONTINUE                                                            PIP01900
C                                                                         PIP01910
C         READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR    PIP01920
```

```
C                                                                       PIP01930
CC          ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE WALL.            PIP01940
CCC                                                                     PIP01950
CCCC        BOUNDARY CONDITIONS AT THE WALL ARE: DQ(1)=0 AND Q(1)=0.    PIP01960
CCCCC                                                                   PIP01970
CC          FOR SECOND EQUATION IN Q(R) ONLY, FOR N = 0.                PIP01980
C                                                                       PIP01990
   55 READ(5,59) I,J,K                                                  PIP02000
      IF(I.EQ.9) GO TO 57                                               PIP02010
      HTEMP = DMESH ** I                                                PIP02020
      JJ = 4 - J                                                        PIP02030
      SIGN = 1.0D0                                                      PIP02040
      IF(I.EQ.1) SIGN = -SIGN                                           PIP02050
      IF(I.EQ.3) SIGN = -SIGN                                           PIP02060
      DO 56 KKK = 1,K                                                   PIP02070
      KK = 7 - KKK                                                      PIP02080
      READ(5,* ) WATEMP                                                 PIP02090
      WATEMP = WATEMP * HTEMP * SIGN                                    PIP02100
      AWALL(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                             PIP02110
   56 CONTINUE                                                          PIP02120
      GO TO 55                                                          PIP02130
   57 CONTINUE                                                          PIP02140
   59 FORMAT(I1,1X,I1,1X,I1)                                            PIP02150
C                                                                       PIP02160
CCC         REARRANGING SOME OF THE DATA IN THE AXIS AND WALL           PIP02170
CCC                                                                     PIP02180
CC          COEFFICIENT MATRICES                                        PIP02190
C                                                                       PIP02200
      DO 58 I = 1,6                                                     PIP02210
      AAXIS(2,3,I) = A2DER(I + 1)                                       PIP02220
      AAXIS(1,3,I) = A1DER(I + 1)                                       PIP02230
      AWALL(2,1,I) = A2DER(I)                                           PIP02240
      AWALL(1,1,I) = A1DER(I)                                           PIP02250
   58 CONTINUE                                                          PIP02260
C                                                                       PIP02270
CC          READ IN COEFFICIENTS TO DETERMINE Q(0)                      PIP02280
C                                                                       PIP02290
      DO 10 I = 1,6                                                     PIP02300
      READ (5,*) A6INV(I)                                               PIP02310
  10  CONTINUE                                                          PIP02320
C                                                                       PIP02330
C*************************************************************************PIP02340
C                                                                       PIP02350
CC          READ IN ALPHA AND REYNOLDS NR                               PIP02360
C                                                                       PIP02370
      IPLOT = 0                                                         PIP02380
   19 CONTINUE                                                          PIP02390
      READ(5,*)  ALPHA,RE                                               PIP02400
```

```
      IF(ALPHA.LT.0.0D0) GO TO 180                                      PIP02410
C                                                                       PIP02420
C*************************************************************        PIP02430
C                                                                       PIP02440
C                              PART III                                 PIP02450
C                                                                       PIP02460
C        COMPUTATION OF THE VORTICITY TRANSPORT EQUATION COEFFICIENTS   PIP02470
C                                                                       PIP02480
C        FOR Q(R) ONLY AND THE RESPECTIVE DERIVATIVES.                  PIP02490
C                                                                       PIP02500
C                                                                       PIP02510
C        PRE-COMPUTE CONSTANTS INVOLVING ALPHA IN THE COEFFICIENTS      PIP02520
C                                                                       PIP02530
      DTEMPR = (-6.0D0/RE)                                              PIP02540
      D3TEMP = DCMPLX(DTEMPR,0.0D0)                                     PIP02550
      E1TEMP = DCMPLX(-3.0D0,0.0D0)                                     PIP02560
      ALSQ = ALPHA * ALPHA                                              PIP02570
      AL2 = ALPHA + ALPHA                                               PIP02580
      AL22 = ALSQ + ALSQ                                                PIP02590
      AL62 = 6.0D0 * ALSQ                                               PIP02600
      AL6 = 6.0D0 * ALPHA                                               PIP02610
      AL3RD = ALSQ * ALPHA                                              PIP02620
      AL4TH = AL3RD * ALPHA                                             PIP02630
      AL23 = AL3RD + AL3RD                                              PIP02640
C                                                                       PIP02650
      DO 60 I = 1,MESH                                                  PIP02660
      R = (DFLOAT(I) - 0.5D0) / DMESH                                   PIP02670
      R2 = R * R                                                        PIP02680
      R3 = R2 * R                                                       PIP02690
      RAD(I) = DCMPLX(R,0.0D0)                                          PIP02700
      DTEMPR = (-R/RE)                                                  PIP02710
      D4(I) = DCMPLX(DTEMPR,0.0D0)                                      PIP02720
      D3(I) = D3TEMP                                                    PIP02730
      DTEMPR = ((-3.0D0/R) + (AL22 * R))/RE                             PIP02740
      DTEMPI = AL2 * (R - R3)                                           PIP02750
      D2(I) = DCMPLX(DTEMPR,DTEMPI)                                     PIP02760
      DTEMPR = ((3.0D0/(R2)) + AL62)/RE                                 PIP02770
      DTEMPI = AL6 * (1.0D0 - R2)                                       PIP02780
      D1(I) = DCMPLX(DTEMPR,DTEMPI)                                     PIP02790
      DTEMPR =-(AL4TH * R)/ RE                                          PIP02800
      DTEMPI = AL23 * (R3 - R)                                          PIP02810
      D0(I) = DCMPLX(DTEMPR,DTEMPI)                                     PIP02820
      ETEMP = -R                                                        PIP02830
      E2(I) = DCMPLX(ETEMP,0.0D0)                                       PIP02840
      E1(I) = E1TEMP                                                    PIP02850
      ETEMP = ALSQ * R                                                  PIP02860
      E0(I) = DCMPLX(ETEMP,0.0D0)                                       PIP02870
   60 CONTINUE                                                          PIP02880
```

```
C                                                                       PIP02890
C***********************************************************************PIP02900
C                                                                       PIP02910
C                                  PART IV                              PIP02920
C                                                                       PIP02930
C        COMPUTATION OF THE A AND B MATRIX ELEMENTS.                    PIP02940
C                                                                       PIP02950
C        INITIALIZE THE A AND B MATRICES                                PIP02960
C                                                                       PIP02970
      DO 65 I = 1,MESH                                                  PIP02980
      DO 65 J = 1,MESH                                                  PIP02990
      A(I,J) = ZERO                                                     PIP03000
      B(I,J) = ZERO                                                     PIP03010
   65 CONTINUE                                                          PIP03020
C                                                                       PIP03030
C        COMPUTATION OF A AND B MATRIX ELEMENTS AT POINTS 1,2,3 (AXIS)  PIP03040
C                                                                       PIP03050
      DO 70 J = 1,3                                                     PIP03060
      DO 70 K = 1,6                                                     PIP03070
      AODER = ZERO                                                      PIP03080
      IF(K.EQ.J) AODER = ONE                                            PIP03090
      A(J,K) = (AAXIS(4,J,K)*D4(J) + AAXIS(3,J,K)*D3(J) +               PIP03100
     1          AAXIS(2,J,K)*D2(J) + AAXIS(1,J,K)*D1(J) + AODER*D0(J))  PIP03110
      B(J,K) = (AAXIS(2,J,K)*E2(J) + AAXIS(1,J,K)*E1(J) + AODER*E0(J))  PIP03120
   70 CONTINUE                                                          PIP03130
C                                                                       PIP03140
C        COMPUTATION OF A AND B MATRIX ELEMENTS AT POINTS 4 - 47        PIP03150
C                                                                       PIP03160
      M = MESH - 3                                                      PIP03170
      DO 80 J = 4,M                                                     PIP03180
      DO 80 K = 1,7                                                     PIP03190
      L = J - 4 + K                                                     PIP03200
      AODER = ZERO                                                      PIP03210
      IF(L.EQ.J) AODER = ONE                                            PIP03220
      A(J,L) = (A4DER(K)*D4(J) + A3DER(K)*D3(J) +                       PIP03230
     1          A2DER(K)*D2(J) + A1DER(K)*D1(J) + AODER*D0(J))          PIP03240
      B(J,L) = (A2DER(K)*E2(J) + A1DER(K)*E1(J) + AODER*E0(J))          PIP03250
   80 CONTINUE                                                          PIP03260
C                                                                       PIP03270
C        COMPUTATION OF A AND B MATRIX ELEMENTS AT POINTS 48 - 50 (WALL)PIP03280
C                                                                       PIP03290
      MM = M - 3                                                        PIP03300
      DO 90 J = 1,3                                                     PIP03310
      L = J + M                                                         PIP03320
      DO 90 K = 1,6                                                     PIP03330
      N = MM + K                                                        PIP03340
      AODER = ZERO                                                      PIP03350
      IF(L.EQ.N) AODER = ONE                                            PIP03360
```

```
      A(L,N) = (AWALL(4,J,K)*D4(L) + AWALL(3,J,K)*D3(L) +                PIP03370
     1          AWALL(2,J,K)*D2(L) + AWALL(1,J,K)*D1(L) + AODER*DO(L))   PIP03380
      B(L,N) = (AWALL(2,J,K)*E2(L) + AWALL(1,J,K)*E1(L) + AODER*EO(L))   PIP03390
   90 CONTINUE                                                           PIP03400
C                                                                        PIP03410
C*************************************************************         PIP03420
C                                                                        PIP03430
C                              PART V                                    PIP03440
C                                                                        PIP03450
C        COMPUTATION OF EIGENVALUES AND RESPECTIVE EIGENVECTORS          PIP03460
C                                                                        PIP03470
C        USING THE SUBROUTINE EIGZC WITH THE VORTICITY TRANSPORT         PIP03480
C                                                                        PIP03490
C        EQUATIONS IN THE FORM  A * X = GAMMA * B * X.                   PIP03500
C                                                                        PIP03510
C                                                                        PIP03520
C             INITIALIZE EIGENVECTOR AND EIGENVALUE ARRAYS.              PIP03530
C                                                                        PIP03540
      DO 140 I = 1,50                                                    PIP03550
      EIGVAL(I) = ZERO                                                   PIP03560
      EIGA(I) = ZERO                                                     PIP03570
      EIGB(I) = ZERO                                                     PIP03580
      DO 140 J = 1,50                                                    PIP03590
      EIGVEC(I,J) = ZERO                                                 PIP03600
      AA(I,J) = A(I,J)                                                   PIP03610
      BB(I,J) = B(I,J)                                                   PIP03620
  140 CONTINUE                                                           PIP03630
      WRITE(6,145)                                                       PIP03640
  145 FORMAT(1X,//,1X,'**************************',                      PIP03650
     1   '**************************************',                      PIP03660
     2   '*********************************************',//)            PIP03670
      IJOB = 1                                                           PIP03680
      CALL EIGZC(AA,50,BB,50,MESH,IJOB,EIGA,EIGB,EIGVEC,50,WK,INEER,IER) PIP03690
      IF(INEER.EQ.0) WRITE(6,150)                                        PIP03700
  150 FORMAT(1X,/////,10X,'ALL EIGENVALUES CONVERGED WITHIN 30 ',        PIP03710
     1       'ITERATIONS',//)                                           PIP03720
      IF(INEER.NE.0) WRITE(6,155) INEER                                  PIP03730
  155 FORMAT(1X,/////,10X,'FAILED TO CONVERGE ON THE',I4,1X,             PIP03740
     1       'TH EIGENVALUE',//)                                        PIP03750
      DO 165 I = 1,MESH                                                  PIP03760
      EIGVAL(I) = EIGA(I)/EIGB(I)                                        PIP03770
      GAMR(I) = DREAL(EIGVAL(I))                                         PIP03780
  165 CONTINUE                                                           PIP03790
      NGAM = 1                                                           PIP03800
      GAMMAX = GAMR(1)                                                   PIP03810
      DO 170 I = 2,MESH                                                  PIP03820
      IF(GAMR(I).LT.GAMMAX) GO TO 169                                    PIP03830
      GAMMAX = GAMR(I)                                                   PIP03840
```

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      NGAM = I                                                          PIP03850
  169 CONTINUE                                                          PIP03860
  170 CONTINUE                                                          PIP03870
      WRITE(6,175) NGAM,GAMMAX,MESH,ALPHA,RE                            PIP03880
  175 FORMAT(1X,/,1X,'LEAST STABLE EIGENVALUE NR',I3,1X,'=',D27.18,4X,  PIP03890
     1 'MESH =',I3,4X,'ALPHA =',D10.3,4X,'REYNOLDS NR =',D17.10,/)      PIP03900
C                                                                       PIP03910
C*********************************************************************  PIP03920
C                                                                       PIP03930
C                              PART VI                                  PIP03940
C                                                                       PIP03950
C        VERIFY THAT THE LEAST STABLE EIGENVALUE AND CORRESPONDING      PIP03960
C                                                                       PIP03970
C        EIGENVECTOR SATISFY THE GOVERNING PARTIAL DIFFERENTIAL         PIP03980
C                                                                       PIP03990
C        EQUATIONS.  RESIDUAL ERROR IS OUTPUT AS EPSILON.               PIP04000
C                                                                       PIP04010
      DO 351 I = 1,MESH                                                 PIP04020
      SUMA = ZERO                                                       PIP04030
      SUMB = ZERO                                                       PIP04040
      DO 350 J = 1,MESH                                                 PIP04050
      SUMA = SUMA + ( A(I,J) * EIGVEC(J,NGAM) )                         PIP04060
      SUMB = SUMB + ( B(I,J) * EIGVEC(J,NGAM) )                         PIP04070
  350 CONTINUE                                                          PIP04080
      ERROR(I) = SUMA - ( SUMB * EIGVAL(NGAM) )                         PIP04090
  351 CONTINUE                                                          PIP04100
      WRITE(6,375)                                                      PIP04110
  375 FORMAT(1X,//,15X,'EPSILON = A * X  -  GAMMA * B * X',/)           PIP04120
      WRITE(6,380) (ERROR(I),I = 1,MESH)                                PIP04130
  380 FORMAT(1X,2D30.18)                                                PIP04140
C                                                                       PIP04150
C*********************************************************************  PIP04160
C                                                                       PIP04170
C                              PART VII                                 PIP04180
C                                                                       PIP04190
C        COMPUTATION OF AXIAL PERTURBATION VELOCITY  U(R).              PIP04200
C                                                                       PIP04210
C                                                                       PIP04220
C        OUTPUT THE EIGENVECTOR CORRESPONDING TO THE LEASE STABLE       PIP04230
C                                                                       PIP04240
C        EIGENVALUE,  GAMMA*.                                           PIP04250
C                                                                       PIP04260
      WRITE(6,398)                                                      PIP04270
  398 FORMAT(1X,///,5X,'EIGENVECTOR Q(I) CORRESPONDING TO THE LEAST ',  PIP04280
     1       'STABLE EIGENVALUE',/)                                     PIP04290
      DO 400 I = 1,MESH                                                 PIP04300
      Q(I) = EIGVEC(I,NGAM)                                             PIP04310
      WRITE(6,399) Q(I)                                                 PIP04320
```

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  399 FORMAT(5X,D30.18,3X,D30.18)                                       PIP04330
  400 CONTINUE                                                          PIP04340
C                                                                       PIP04350
CC      COMPUTATION OF DQ(R)                                            PIP04360
C                                                                       PIP04370
      DO 410 J = 1,3                                                    PIP04380
      DQ(J) = ZERO                                                      PIP04390
      DO 410 K = 1,6                                                    PIP04400
      DQ(J) = DQ(J) + Q(K) * AAXIS(1,J,K)                               PIP04410
  410 CONTINUE                                                          PIP04420
C                                                                       PIP04430
      M = MESH - 3                                                      PIP04440
      DO 420 J = 4,M                                                    PIP04450
      DQ(J) = ZERO                                                      PIP04460
      DO 420 K = 1,7                                                    PIP04470
      L = J - 4 + K                                                     PIP04480
      DQ(J) = DQ(J) + Q(L) * A1DER(K)                                   PIP04490
  420 CONTINUE                                                          PIP04500
C                                                                       PIP04510
      MM = M - 3                                                        PIP04520
      DO 430 J = 1,3                                                    PIP04530
      L = J + M                                                         PIP04540
      DQ(L) = ZERO                                                      PIP04550
      DO 430 K = 1,6                                                    PIP04560
      N = MM + K                                                        PIP04570
      DQ(L) = DQ(L) + Q(N) * AWALL(1,J,K)                               PIP04580
  430 CONTINUE                                                          PIP04590
C                                                                       PIP04600
      DO 440 I = 1,MESH                                                 PIP04610
      U(I) = Q(I) * TWO + RAD(I) * DQ(I)                                PIP04620
  440 CONTINUE                                                          PIP04630
C                                                                       PIP04640
CC        COMPUTE Q(0)                                                  PIP04650
C                                                                       PIP04660
      Q06 = ZERO                                                        PIP04670
      DO 441 I = 1,6                                                    PIP04680
      A6CPX = DCMPLX(A6INV(I),0.0D0)                                    PIP04690
      Q06 = Q06 + ( A6CPX * Q(I) )                                      PIP04700
  441 CONTINUE                                                          PIP04710
C                                                                       PIP04720
CC        REARRANGE DATA FOR PLOT ROUTINE                               PIP04730
C                                                                       PIP04740
      M2 = MESH + 2                                                     PIP04750
      DO 450 I = 1,MESH                                                 PIP04760
      II = M2 - I                                                       PIP04770
      III = II - 1                                                      PIP04780
      U(II) = U(III)                                                    PIP04790
      RAD(II) = RAD(III)                                                PIP04800
```

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  450 CONTINUE                                                          PIP04810
      RAD(1) = ZERO                                                     PIP04820
      RAD(M2) = ONE                                                     PIP04830
      U(1) = Q06 * TWO                                                  PIP04840
      U(M2) = ZERO                                                      PIP04850
      DO 460 I = 1,M2                                                   PIP04860
      RADR(I) = DREAL(RAD(I))                                           PIP04870
  460 CONTINUE                                                          PIP04880
C                                                                       PIP04890
CC        COMPUTE THE NORMALIZED PERTURBATION VELOCITY ANDAMPLITUDE     PIP04900
C                                                                       PIP04910
      DO 465 I = 1,M2                                                   PIP04920
      UAMP(I) = CDABS(U(I))                                             PIP04930
  465 CONTINUE                                                          PIP04940
      AMPMAX = UAMP(1)                                                  PIP04950
      IMAX = 1                                                          PIP04960
      DO 470 J = 2,M2                                                   PIP04970
      AMTEMP = UAMP(J)                                                  PIP04980
      IF(AMTEMP.LT.AMPMAX) GO TO 471                                    PIP04990
      AMPMAX = UAMP(J)                                                  PIP05000
      IMAX = J                                                          PIP05010
  471 CONTINUE                                                          PIP05020
  470 CONTINUE                                                          PIP05030
      WRITE(6,472)                                                      PIP05040
  472 FORMAT(1X,///,23X,'NORMALIZED PERTURBATION VELOCITY',             PIP05050
     124X,'NORMALIZED AMPLITUDE',/)                                     PIP05060
      DO 475 K = 1,M2                                                   PIP05070
      UAMNOR(K) = UAMP(K) / AMPMAX                                      PIP05080
      AMPNOR(K) = UAMNOR(K)                                             PIP05090
      UNORM(K) = U(K) / U(IMAX)                                         PIP05100
      URDP(K) = DREAL(UNORM(K))                                         PIP05110
      UR(K) = URDP(K)                                                   PIP05120
      UIDP(K) = DIMAG(UNORM(K))                                         PIP05130
      UI(K) = UIDP(K)                                                   PIP05140
      KK = K - 1                                                        PIP05150
      WRITE(6,466) KK,URDP(K),JIDP(K),AMPNOR(K)                         PIP05160
  466 FORMAT(1X,I3,1X,2D30.18,5X,D30.18)                                PIP05170
  475 CONTINUE                                                          PIP05180
C                                                                       PIP05190
CC*****************************************************************   PIP05200
C                                                                       PIP05210
C                                 PART VIII                             PIP05220
C                                                                       PIP05230
CC        PLOT THE NORMALIZED PERTURBATION VELOCITY VS PIPE RADIUS.     PIP05240
C                                                                       PIP05250
      IPLOT = IPLOT + 1                                                 PIP05260
      RMESH = FLOAT(MESH)                                               PIP05270
      ALPLOT = ALPHA                                                    PIP05280
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      REPLOT = RE                                                       PIP05290
      GAMPLT = GAMMAX                                                   PIP05300
      CALL PLOTG(RADR,UR,M2,1,1,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0) PIP05310
      CALL PLOTG(RADR,UI,M2,2,1,5,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0) PIP05320
C     CALL PLOTG(RADR,AMPNDR,M2,3,1,2,' ',1,' ',1,0.,1.,-1.,1.,5.,5.)   PIP05330
      CALL PLOTG(X1,Y1,2,4,1,0,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)    PIP05340
      XO = 0.15                                                         PIP05350
      YO = 5.6                                                          PIP05360
      IF(IPLOT.EQ.4) IPLOT = 1                                          PIP05370
      IF(IPLOT.EQ.2) YO = 12.1                                          PIP05380
      IF(IPLOT.EQ.3) YO = 18.6                                          PIP05390
      CALL TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                     PIP05400
      GO TO 19                                                          PIP05410
  180 CONTINUE                                                          PIP05420
      CALL PLOT(0.0,0.0,999)                                            PIP05430
      STOP                                                              PIP05440
      END                                                               PIP05450
C                                                                       PIP05460
CC    ..............................................................    PIP05470
CC                                                                      PIP05480
CC    SUBROUTINE TITLE1                                                 PIP05490
CC                                                                      PIP05500
CC       SUBROUTINE TO WRITE TITLE ON PLOTS                             PIP05510
C                                                                       PIP05520
      SUBROUTINE TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)               PIP05530
      RN = 0.0                                                          PIP05540
      YHT = 0.14                                                        PIP05550
      CALL NEWPEN(2)                                                    PIP05560
      CALL SYMBOL(XO,YO,YHT,                                            PIP05570
     1'NORMALIZED PERTURBATION VELOCITY VS RADIUS',0.0,42)              PIP05580
      YHT = 0.08                                                        PIP05590
      DELY = 0.20                                                       PIP05600
      CALL NEWPEN(1)                                                    PIP05610
      YO = YO - 0.8                                                     PIP05620
      XO = XO + 2.5                                                     PIP05630
      XN = XO + .75                                                     PIP05640
      CALL SYMBOL(XO,YO,YHT,'N       = ',0.0,9)                         PIP05650
      CALL NUMBER(XN,YO,YHT,RN,0.0,-1)                                  PIP05660
      YO = YO - DELY                                                    PIP05670
      CALL SYMBOL(XO,YO,YHT,'MESH    = ',0.0,9)                         PIP05680
      CALL NUMBER(XN,YO,YHT,RMESH,0.0,-1)                               PIP05690
      YO = YO - DELY                                                    PIP05700
      CALL SYMBOL(XO,YO,YHT,'ALPHA   = ',0.0,9)                         PIP05710
      CALL NUMBER(XN,YO,YHT,ALPLOT,0.0,2)                               PIP05720
      YO = YO - DELY                                                    PIP05730
      CALL SYMBOL(XO,YO,YHT,'REY NR  = ',0.0,9)                         PIP05740
      CALL NUMBER(XN,YO,YHT,REPLOT,0.0,1)                               PIP05750
      YO = YO - DELY                                                    PIP05760
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      CALL SYMBOL(XO,YO,YHT,'GAMMA* = ',0.0,9)                                  PIP05770
      CALL NUMBER(XN,YO,YHT,GAMPLT,0.0,4)                                       PIP05780
      YO = YO - 3.4                                                             PIP05790
      CALL SYMBOL(XO,YO,YHT,'OCTAGON = U(REAL)',0.0,17)                         PIP05800
      YO = YO - DELY                                                            PIP05810
      CALL SYMBOL(XO,YO,YHT,'DIAMOND = U(IMAG)',0.0,17)                         PIP05820
      YO = YO - DELY                                                            PIP05830
C     CALL SYMBOL(XO,YO,YHT,'TRIANGLE= U AMPL ',0.0,17)                         PIP05840
C                                                                               PIP05850
CC       WRITE THE AXIS LABLES                                                  PIP05860
C                                                                               PIP05870
      XO = XO - 2.55                                                            PIP05880
      YO = YO + 0.3                                                             PIP05890
      YHT = 0.12                                                                PIP05900
      CALL SYMBOL(XO,YO,YHT,'NORMALIZED PERTURBATION VELOCITY',90.,32)          PIP05910
      XO = XO + 2.8                                                             PIP05920
      YO = YO - 1.0                                                             PIP05930
      CALL SYMBOL(XO,YO,YHT,'PIPE RADIUS',0.0,11)                               PIP05940
      RETURN                                                                    PIP05950
      END                                                                       PIP05960
/*                                                                              PIP05970
//GO.SYSIN DD *                                                                 PIP05980
4 1 6                                                                           PIP05990
      0.14905527754213541 1D+01                                                 PIP06000
     -0.14672855130067858 9D+01                                                 PIP06010
     -0.95280803446403548 9D+00                                                 PIP06020
      0.13147481450175468 9D+01                                                 PIP06030
     -0.44306626816295204 0D+00                                                 PIP06040
      0.57858895194873314 1D-01                                                 PIP06050
4 2 6                                                                           PIP06060
     -0.52997087407008978 1D+01                                                 PIP06070
      0.10923281172878383 0D+02                                                 PIP06080
     -0.80354336201689449 1D+01                                                 PIP06090
      0.27666474282402671 0D+01                                                 PIP06100
     -0.38098574438380006 2D+00                                                 PIP06110
      0.26199504134987103 3D-01                                                 PIP06120
4 3 6                                                                           PIP06130
      0.18299535726319717 3D+01                                                 PIP06140
     -0.64925902480678203 7D+01                                                 PIP06150
      0.93268795274362297 2D+01                                                 PIP06160
     -0.64967929150805332 5D+01                                                 PIP06170
      0.19991083579918473 8D+01                                                 PIP06180
     -0.16655829491175344 6D+00                                                 PIP06190
3 1 5                                                                           PIP06200
      0.30710207612456812 4D+01                                                 PIP06210
     -0.55170415224913651 5D+01                                                 PIP06220
      0.31907439446366840 9D+01                                                 PIP06230
     -0.84801038062284062 0D+00                                                 PIP06240
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  0.10328719723183425 7D+00                                          PIP06250
3 2 5                                                                PIP06260
  0.90960207612456422 9D+00                                          PIP06270
 -0.44420415224913223 4D+00                                          PIP06280
 -0.12184256055363442 7D+01                                          PIP06290
  0.85769896193771688 6D+00                                          PIP06300
 -0.10467128027681674 2D+00                                          PIP06310
3 3 5                                                                PIP06320
 -0.29096020761245817 7D+01                                          PIP06330
  0.64442041522491599 9D+01                                          PIP06340
 -0.47815743944636999 2D+01                                          PIP06350
  0.11423010380622855 5D+01                                          PIP06360
  0.10467128027681571 5D+00                                          PIP06370
2 1 4                                                                PIP06380
 -0.12228993261989693 5D+01                                          PIP06390
  0.13512187871581455 8D+01                                          PIP06400
 -0.13956599286563649 3D+00                                          PIP06410
  0.11246531906460608 5D-01                                          PIP06420
2 2 4                                                                PIP06430
  0.12497522790328994 3D+01                                          PIP06440
 -0.24995541022592189 2D+01                                          PIP06450
  0.13330856123662322 4D+01                                          PIP06460
 -0.83283789139913064 2D-01                                          PIP06470
1 1 3                                                                PIP06480
 -0.66666666666666668 5D+00                                          PIP06490
  0.75000000000000000 0D+00                                          PIP06500
 -0.83333333333333314 8D-01                                          PIP06510
1 2 3                                                                PIP06520
 -0.99999999999999977 8D+00                                          PIP06530
  0.74999999999999977 8D+00                                          PIP06540
  0.25000000000000013 9D+00                                          PIP06550
9 0 0                                                                PIP06560
4 1 6                                                                PIP06570
  0.11500000000000071 4D+03                                          PIP06580
 -0.38444444444445025 5D+02                                          PIP06590
  0.16400000000000385 8D+02                                          PIP06600
 -0.52244897959184979 4D+01                                          PIP06610
  0.11234567901234950 9D+01                                          PIP06620
 -0.11570247933884755 8D+00                                          PIP06630
4 2 6                                                                PIP06640
 -0.26000000000000802 9D+02                                          PIP06650
  0.17666666666567016 0D+02                                          PIP06660
 -0.11200000000000230 2D+02                                          PIP06670
  0.39591836734695107 2D+01                                          PIP06680
 -0.66666666666670049 1D+00                                          PIP06690
  0.57851239669425180 7D-01                                          PIP06700
4 3 6                                                                PIP06710
  0.10000000000008526 5D+01                                          PIP06720
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      -0.62222222222280408D+01                                  PIP06730
       0.92000000000045759D+01                                  PIP06740
      -0.64489795918368990D+01                                  PIP06750
       0.19876543209876977 9D+01                                PIP06760
      -0.16528925619835255 2D+00                                PIP06770
3 1 5                                                           PIP06780
      -0.17499999999999975 1D+02                                PIP06790
       0.11111111111107603D+00                                  PIP06800
       0.12000000000001372D+01                                  PIP06810
      -0.34693877551020739 5D+00                                PIP06820
       0.43209876543210391 1D-01                                PIP06830
3 2 5                                                           PIP06840
       0.13499999999999968 0D+02                                PIP06850
      -0.38888888888888821 3D+01                                PIP06860
      -0.14210854715202003 7D-13                                PIP06870
       0.55102040816326658 4D+00                                PIP06880
      -0.67901234567901785 9D-01                                PIP06890
3 3 5                                                           PIP06900
      -0.15500000000000113 7D+02                                PIP06910
       0.98888888888891060D+01                                  PIP06920
      -0.60000000000000532 9D+01                                PIP06930
       0.14489795918367498 5D+01                                PIP06940
       0.67901234567899093 6D-01                                PIP06950
2 1 4                                                           PIP06960
      -0.46666666666666585 0D+01                                PIP06970
       0.21111111111111102 7D+01                                PIP06980
      -0.31999999999999993 0D+00                                PIP06990
       0.34013605442176852 1D-01                                PIP07000
2 2 4                                                           PIP07010
       0.99999999999978684 0D+00                                PIP07020
      -0.24444444444444493 0D+01                                PIP07030
       0.13200000000000011 7D+01                                PIP07040
      -0.81632653061224511 0D-01                                PIP07050
1 1 3                                                           PIP07060
       0.24999999999999988 9D+01                                PIP07070
       0.22222222222222229 3D+00                                PIP07080
      -0.19999999999999978 0D-01                                PIP07090
1 2 3                                                           PIP07100
      -0.45000000000000035 5D+01                                PIP07110
       0.13333333333333304 0D+01                                PIP07120
       0.18000000000000007 0D+00                                PIP07130
9 0 0                                                           PIP07140
       0.12791408217392929 0D+01                                PIP07150
      -0.41669653599285105 8D+00                                PIP07160
       0.19554923111095109 0D+00                                PIP07170
      -0.73690837175280396 5D-01                                PIP07180
       0.17653187827415903 9D-01                                PIP07190
      -0.19558675041653811 8D-02                                PIP07200
```

```
0.0D0     575.0D0                                                    PIP07210
0.5D0     575.0D0                                                    PIP07220
1.0D0     575.0D0                                                    PIP07230
2.0D0     575.0D0                                                    PIP07240
4.0D0     575.0D0                                                    PIP07250
8.0D0     575.0D0                                                    PIP07260
16.0D0    575.0D0                                                    PIP07270
32.0D0    575.0D0                                                    PIP07280
0.0D0     1150.0D0                                                   PIP07290
0.5D0     1150.0D0                                                   PIP07300
1.0D0     1150.0D0                                                   PIP07310
2.0D0     1150.0D0                                                   PIP07320
4.0D0     1150.0D0                                                   PIP07330
8.0D0     1150.0D0                                                   PIP07340
16.0D0    1150.0D0                                                   PIP07350
32.0D0    1150.0D0                                                   PIP07360
-1.0D0 1.0D0                                                         PIP07370
/*                                                                   PIP07380
//                                                                   PIP07390
```

```
//WALLACE JOB (2027,0084),'THESIS N = 1',CLASS=C                    PIP00010
//*MAIN LINES=(10)                                                  PIP00020
// EXEC FRTXCLGP,IMSL=DP,REGION.GO=2048K                            PIP00030
//FORT.SYSIN DD *                                                   PIP00040
C                                                                   PIP00050
C*******************************************************************PIP00060
C                                                                   PIP00070
C       THIS PROGRAM WAS DEVELOPED TO PERFORM A NUMERICAL ANALYSIS  PIP00080
C       OF THREE DIMENSIONAL PIPE FLOW STABILITY FOR AXIAL WAVE     PIP00090
C       NUMBER N = 1. THE PROGRAM IS DIVIDED INTO EIGHT PARTS.      PIP00100
C       THE FIRST PART COMPUTES THE CENTRAL FINITE DIFFERENCE       PIP00110
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT THE    PIP00120
C       INTERIOR MESH POINTS ALONG THE PIPE RADIUS. IN THE SECOND   PIP00130
C       PART,  PRE-COMPUTED NON-CENTRAL FINITE DIFFERENCE           PIP00140
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT POINTS PIP00150
C       NEAR THE AXIS AND WALL ARE READ IN AS DATA. THE THIRD PART  PIP00160
C       COMPUTES THE VORTICITY TRANSPORT EQUATION COEFFICIENTS OF   PIP00170
C       P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES AT ALL RADIAL PIP00180
C       MESH POINTS (STATIONS). THE FOURTH PART COMPUTES THE A AND B PIP00190
C       MATRIX ELEMENTS,  WHICH MAKE-UP THE EIGENSYSTEM TO BE SOLVED PIP00200
C       IN THE FORMAT,  A * X = GAMMA * B * X.  IN THE FIFTH PART   PIP00210
C       THE A AND B MATRICES COMPRISING THE EIGENSYSTEM ARE SOLVED  PIP00220
C       IN THE SUBROUTINE EIGZC.  THE RESULTING EIGENVALUES AND     PIP00230
C       CORRESPONDING EIGENVECTOR REPRESENT GAMMA AND THE COLUMN    PIP00240
C       VECTOR X,  RESPECTIVELY, AND ARE SOLUTIONS OF THE GOVERNING PIP00250
C       PARTIAL DIFFERENTIAL EQUATIONS.  THE SIXTH PART VERIFIES    PIP00260
C       THE SOLUTION OF THE PDE AND THE MAGNITUDE OF THE RESIDUAL   PIP00270
C       IS DETERMINED.  THE SEVENTH PART COMPUTES THE NORMALIZED    PIP00280
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C
CC          AXIAL PERTURBATION VELOCITY U(R).  THE EIGHTH AND LAST          PIP00490
CC          PART PLOTS THE PERTURBATION VELOCITY VERSUS PIPE RADIUS.        PIP00500
CC          REQUIRED SUBROUTINES ARE INCLUDED AT THE END OF THE PROGRAM     PIP00510
CC          LISTING.  THIS IS A GENERALIZED PROGRAM FOR VARIABLE SIZED      PIP00520
CC          MESH UP TO A MAXIMUM MESH SIZE OF 48.  BECAUSE OF THE           PIP00530
CC          LARGE AMOUNT OF MEMORY REQUIRED TO RUN THIS PROGRAM,  IT        PIP00540
CC          CAN ONLY BE RUN IN THE BATCH MODE (MVS).  AVERAGE CPU TIME      PIP00550
CC          REQUIRED FOR ONE SET OF DATA,  I.E. ONE AXIAL WAVE NUMBER       PIP00560
CC          AND ONE REYNOLDS NUMBER,  IS 58 SECONDS.                        PIP00570
CC*****************************************************************      PIP00580
CC                                                                          PIP00590
CC          DEFINE PROGRAM VARIABLES AND ARRAYS.                            PIP00600
C                                                                           PIP00610
      COMPLEX*16 A(100,100),B(100,100)                                      PIP00620
      COMPLEX*16 ZERO,ONE,AAXISQ(4,3,6),AWALLQ(4,3,6),A1DERQ(7)             PIP00630
      COMPLEX*16 A2DERQ(7),A3DERQ(7),A4DERQ(7),A0DER                        PIP00640
      COMPLEX*16 D3Q1(50),D2Q1(50),D1Q1(50),D0Q1(50)                        PIP00650
      COMPLEX*16 D4Q2(50),D3Q2(50),D2Q2(50),D1Q2(50),D0Q2(50)               PIP00660
      COMPLEX*16 E1Q1(50),E0Q1(50),E2Q2(50),E1Q2(50),E0Q2(50)               PIP00670
      COMPLEX*16 AAXISP(2,2,5),AWALLP(2,2,5),A1DERP(5),A2DERP(5)            PIP00680
      COMPLEX*16 D2P1(50),D1P1(50),D0P1(50),D2P2(50),D1P2(50)               PIP00690
      COMPLEX*16 D0P2(50),E0P1(50),E0P2(50)                                 PIP00700
      COMPLEX*16 EIGA(100),EIGB(100),EIGVEC(100,100),EIGVAL(100)            PIP00710
      COMPLEX*16 AA(100,100),BB(100,100),ONEIMG,TWO,THREE                   PIP00720
      COMPLEX*16 Q(50),DQ(50),P(50),J(52),RAD(52),UNORM(52)                 PIP00730
      COMPLEX*16 AI5P(6),AI5D2P(6),AI6D2Q(6),H0A1(52),H0A2(52)              PIP00740
      COMPLEX*16 H0B1(52),H0B2(52),H0,Q0                                    PIP00750
      COMPLEX*16 H0D1PA,H0D2PA,Q0D1QA,Q0D2QA,Q0D3QA,Q0D1QB,BETA             PIP00760
      COMPLEX*16 Q0D4QA,Q0D2QB,Q0TEM1,Q0TEM2,H0D1QA,H0D2QA,H0D3QA           PIP00770
      COMPLEX*16 H0D4QA,H0D1QB,H0D2QB,AAXQ1,AAXQ2,AWAQ1,AWAQ2               PIP00780
      COMPLEX*16 D2P01,D2P02,D2Q01,D2Q02,D0P01,D0P02,D0Q01,D0Q02            PIP00790
      COMPLEX*16 D0H01,D0H02,E0P01,E0P02,E0Q01,E0Q02,E0H01,E0H02            PIP00800
      COMPLEX*16 ERROR(100),WK(1,1),SUMA,SUMB                               PIP00810
      REAL*8 GAMR(100),GAMMAX,DMESH                                         PIP00820
      REAL*8 H4TH,H3RD,H2ND,H1ST,HTEMP,ATEMP,AATEMP,WATEMP,R,RE,ALPHA       PIP00830
      REAL*8 TEMPR,TEMPI,SIGN,R2,R3,R4                                      PIP00840
      REAL*8 AL2,AL4,AL10,ALSQ,AL4TH,ALSQ2,ALSQ4,ALSQ5,ALSQ10,AL3RD2        PIP00850
      REAL*8 UAMP(52),AMPMAX,AMTEMP,UAMNOR(52),URDP(52),UIDP(52)            PIP00860
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      REAL*4 RADR(52),UR(52),UI(52),AMPNOR(52)                          PIP00970
      REAL*4 X1(2),Y1(2),ALPLOT,REPLOT,GAMPLT,RMESH                     PIP00980
      INTEGER MESH,M,MM,M2,IMAX,MP,MMP,MESH2,MH0,MQ0,NGAM               PIP00990
C                                                                       PIP01000
CC        INITIALIZE THE COEFFICIENT ARRAYS AND SEVERAL CONSTANTS       PIP01010
C                                                                       PIP01020
      DATA ZERO/(0.0D0,0.0D0)/,ONE/(1.0D0,0.0D0)/,TWO/(2.0D0,0.0D0)/    PIP01030
      DATA ONEIMG/(0.0D0,1.0D0)/                                        PIP01040
      DATA THREE/(3.0D0,0.0D0)/                                         PIP01050
      DATA AAXISQ/72*(0.0D0,0.0D0)/,AWALLQ/72*(0.0D0,0.0D0)/            PIP01060
      DATA A1DERQ/7*(0.0D0,0.0D0)/,A2DERQ/7*(0.0D0,0.0D0)/              PIP01070
      DATA A3DERQ/7*(0.0D0,0.0D0)/,A4DERQ/7*(0.0D0,0.0D0)/              PIP01080
      DATA AAXISP/20*(0.0D0,0.0D0)/,AWALLP/20*(0.0D0,0.0D0)/            PIP01090
      DATA A1DERP/5*(0.0D0,0.0D0)/,A2DERP/5*(0.0D0,0.0D0)/              PIP01100
      DATA X1(1)/0.0/,X1(2)/1.0/,Y1(1)/0.0/,Y1(2)/0.0/                  PIP01110
      DATA MESH/48/                                                     PIP01120
C                                                                       PIP01130
CC*************************************************************        PIP01140
CCC                                                                     PIP01150
CCC                          PART I                                     PIP01160
CCC                                                                     PIP01170
CCC       COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FOURTH ORDER      PIP01180
CCCC                                                                    PIP01190
CCC       DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                   PIP01200
C                                                                       PIP01210
      DMESH = DFLOAT(MESH)                                              PIP01220
      H4TH = DMESH ** 4                                                 PIP01230
      ATEMP = (-1.0D0/6.0D0) * H4TH                                     PIP01240
      A4DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01250
      A4DERQ(7) = A4DERQ(1)                                             PIP01260
      ATEMP = 2.0D0 * H4TH                                              PIP01270
      A4DERQ(2) = DCMPLX(ATEMP,0.0D0)                                   PIP01280
      A4DERQ(6) = A4DERQ(2)                                             PIP01290
      ATEMP = -6.5D0 * H4TH                                             PIP01300
      A4DERQ(3) = DCMPLX(ATEMP,0.0D0)                                   PIP01310
      A4DERQ(5) = A4DERQ(3)                                             PIP01320
      ATEMP = (56.0D0/6.0D0) * H4TH                                     PIP01330
      A4DERQ(4) = DCMPLX(ATEMP,0.0D0)                                   PIP01340
C                                                                       PIP01350
CC        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THIRD ORDER       PIP01360
CC                                                                      PIP01370
CC        DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                   PIP01380
C                                                                       PIP01390
      H3RD = DMESH ** 3                                                 PIP01400
      ATEMP = 0.125D0 * H3RD                                            PIP01410
      A3DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01420
      A3DERQ(7) = -A3DERQ(1)                                            PIP01430
      A3DERQ(6) = DCMPLX(H3RD,0.0D0)                                    PIP01440
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      A3DERQ(2)  = -A3DERQ(5)                                          PIP01450
      ATEMP = (13.0D0/8.0D0) * H3RD                                    PIP01460
      A3DERQ(3)  = DCMPLX(ATEMP,0.0D0)                                 PIP01470
      A3DERQ(5)  = -A3DERQ(3)                                          PIP01480
C                                                                      PIP01490
CCCC     COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR SECOND ORDER      PIP01500
CCCC                                                                   PIP01510
CCCC     DERIVATIVES OF P(R) AND Q(R) AT INTERIOR MESH POINTS.         PIP01520
C                                                                      PIP01530
      H2ND = DMESH ** 2                                                PIP01540
      ATEMP = (-1.0D0/12.0D0) * H2ND                                   PIP01550
      A2DERQ(2)  = DCMPLX(ATEMP,0.0D0)                                 PIP01560
      A2DERQ(6)  = A2DERQ(2)                                           PIP01570
      ATEMP = (16.0D0/12.0D0) * H2ND                                   PIP01580
      A2DERQ(3)  = DCMPLX(ATEMP,0.0D0)                                 PIP01590
      A2DERQ(5)  = A2DERQ(3)                                           PIP01600
      ATEMP = -2.5D0 * H2ND                                            PIP01610
      A2DERQ(4)  = DCMPLX(ATEMP,0.0D0)                                 PIP01620
      A2DERP(1)  = A2DERQ(2)                                           PIP01630
      A2DERP(2)  = A2DERQ(3)                                           PIP01640
      A2DERP(3)  = A2DERQ(4)                                           PIP01650
      A2DERP(4)  = A2DERQ(5)                                           PIP01660
      A2DERP(5)  = A2DERQ(5)                                           PIP01670
C                                                                      PIP01680
CCCC     COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FIRST ORDER       PIP01690
CCCC                                                                   PIP01700
CCCC     DERIVATIVES OF P(R) AND Q(R) AT INTERIOR MESH POINTS.         PIP01710
C                                                                      PIP01720
      H1ST = DMESH                                                     PIP01730
      ATEMP = (1.0D0/12.0D0) * H1ST                                    PIP01740
      A1DERQ(2)  = DCMPLX(ATEMP,0.0D0)                                 PIP01750
      A1DERQ(6)  = -A1DERQ(2)                                          PIP01760
      ATEMP = (-8.0D0/12.0D0) * H1ST                                   PIP01770
      A1DERQ(3)  = DCMPLX(ATEMP,0.0D0)                                 PIP01780
      A1DERQ(5)  = -A1DERQ(3)                                          PIP01790
      A1DERP(1)  = A1DERQ(2)                                           PIP01800
      A1DERP(2)  = A1DERQ(3)                                           PIP01810
      A1DERP(4)  = A1DERQ(5)                                           PIP01820
      A1DERP(5)  = A1DERQ(5)                                           PIP01830
C                                                                      PIP01840
C**********************************************************************PIP01850
C                                                                      PIP01860
CCCC                          PART II                                  PIP01870
CCCC                                                                   PIP01880
CCCC     READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP01890
CCCC                                                                   PIP01900
CCCC     ALL DERIVATIVES OF P(R) AT POINTS NEAR THE AXIS.              PIP01910
C                                                                      PIP01920
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C          BOUNDARY CONDITIONS AT THE AXIS ARE: DP(0) = 0 AND Q(0) = VAR.  PIP01930
C                                                                          PIP01940
   41 READ(5,49) I,J,K                                                     PIP01950
      IF(I.EQ.9) GO TO 44                                                  PIP01960
      HTEMP = DMESH ** I                                                   PIP01970
      DO 42 KKK = 1,K                                                      PIP01980
      READ(5,*) AATEMP                                                     PIP01990
      AATEMP = AATEMP * HTEMP                                              PIP02000
      AAXISP(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                               PIP02010
   42 CONTINUE                                                             PIP02020
      GO TO 41                                                             PIP02030
   44 CONTINUE                                                             PIP02040
C                                                                          PIP02050
CCC        READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS        PIP02060
CCCC                                                                       PIP02070
CCC        FOR P(0) AND D2P(J).                                            PIP02080
C                                                                          PIP02090
      DO 70 I = 1,5                                                        PIP02100
      READ(5,*) AATEMP                                                     PIP02110
      AI5P(I) = DCMPLX(AATEMP,0.0D0)                                       PIP02120
   70 CONTINUE                                                             PIP02130
      AI5P(6) = ZERO                                                       PIP02140
      DO 71 I = 1,5                                                        PIP02150
      READ(5,*) AATEMP                                                     PIP02160
      AATEMP = AATEMP * (DMESH ** 2)                                       PIP02170
      AI5D2P(I) = DCMPLX(AATEMP,0.0D0)                                     PIP02180
   71 CONTINUE                                                             PIP02190
      AI5D2P(6) = ZERO                                                     PIP02200
C                                                                          PIP02210
CCC        READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR    PIP02220
CCCC                                                                       PIP02230
CCC        ALL DERIVATIVES OF P(R) AT POINTS NEAR THE WALL.                PIP02240
CCC                                                                        PIP02250
CCC        BOUNDARY CONDITION AT THE WALL IS:  P(1) = I * H(0)             PIP02260
C                                                                          PIP02270
   45 READ(5,49) I,J,K                                                     PIP02280
      IF(I.EQ.9) GO TO 47                                                  PIP02290
      HTEMP = DMESH ** I                                                   PIP02300
      JJ = 3 - J                                                           PIP02310
      SIGN = 1.0D0                                                         PIP02320
      IF(I.EQ.1) SIGN = -SIGN                                              PIP02330
      DO 46 KKK = 1,K                                                      PIP02340
      KK = 6 - KKK                                                         PIP02350
      READ(5,*) WATEMP                                                     PIP02360
      WATEMP = WATEMP * HTEMP * SIGN                                       PIP02370
      AWALLP(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                               PIP02380
   46 CONTINUE                                                             PIP02390
      GO TO 45                                                             PIP02400
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   47 CONTINUE                                                          PIP02410
   49 FORMAT(I1, 1X, I1, 1X, I1)                                        PIP02420
C                                                                       PIP02430
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR   PIP02440
CCC                                                                     PIP02450
CCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE AXIS.               PIP02460
CCC                                                                     PIP02470
CCC      BOUNDARY CONDITION AT THE AXIS IS: DQ(0) = 0.                  PIP02480
C                                                                       PIP02490
   51 READ(5,49) I,J,K                                                  PIP02500
      IF(I.EQ.9) GO TO 54                                               PIP02510
      HTEMP = DMESH ** I                                                PIP02520
      DO 52 KKK = 1,K                                                   PIP02530
      READ(5,*) AATEMP                                                  PIP02540
      AATEMP = AATEMP * HTEMP                                           PIP02550
      AAXISQ(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                            PIP02560
   52 CONTINUE                                                          PIP02570
      GO TO 51                                                          PIP02580
   54 CONTINUE                                                          PIP02590
C                                                                       PIP02600
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS       PIP02610
CCC                                                                     PIP02620
CCC      FOR  D2Q(0).                                                   PIP02630
C                                                                       PIP02640
      DO 72 I = 1,6                                                     PIP02650
      READ(5,*) AATEMP                                                  PIP02660
      AATEMP = AATEMP * (DMESH ** 2)                                    PIP02670
      AI6D2Q(I) = DCMPLX(AATEMP,0.0D0)                                  PIP02680
   72 CONTINUE                                                          PIP02690
C                                                                       PIP02700
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR   PIP02710
CCC                                                                     PIP02720
CCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE WALL.               PIP02730
CCC                                                                     PIP02740
CCC      BOUNDARY CONDITIONS AT THE WALL ARE: DQ(1) = 2 * H(0) AND      PIP02750
CCC                                                                     PIP02760
CCC      Q(1) = - H(0).                                                 PIP02770
C                                                                       PIP02780
   55 READ(5,49) I,J,K                                                  PIP02790
      IF(I.EQ.9) GO TO 57                                               PIP02800
      HTEMP = DMESH ** I                                                PIP02810
      JJ = 4 - J                                                        PIP02820
      SIGN = 1.0D0                                                      PIP02830
      IF(I.EQ.1) SIGN = -SIGN                                           PIP02840
      IF(I.EQ.3) SIGN = -SIGN                                           PIP02850
      DO 56 KKK = 1,K                                                   PIP02860
      KK = 7 - KKK                                                      PIP02870
      READ(5,*) WATEMP                                                  PIP02880
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```
      WATEMP = WATEMP * HTEMP * SIGN                                    PIP02890
      AWALLQ(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                            PIP02900
   56 CONTINUE                                                          PIP02910
      GO TO 55                                                          PIP02920
   57 CONTINUE                                                          PIP02930
C                                                                       PIP02940
CC       REARRANGE SOME OF THE DATA FOR DQ(R) AND D2Q(R) IN THE         PIP02950
CC                                                                      PIP02960
CC       AAXISQ AND AWALLQ COEFFICIENT MATRICES.                        PIP02970
C                                                                       PIP02980
      DO 58 I = 1,6                                                     PIP02990
      AAXISQ(2,3,I) = A2DERQ(I + 1)                                     PIP03000
      AAXISQ(1,3,I) = A1DERQ(I + 1)                                     PIP03010
      AWALLQ(2,1,I) = A2DERQ(I)                                         PIP03020
      AWALLQ(1,1,I) = A1DERQ(I)                                         PIP03030
   58 CONTINUE                                                          PIP03040
C                                                                       PIP03050
CC       READ IN ALPHA AND REYNOLDS NR                                  PIP03060
C                                                                       PIP03070
      IPLOT = 0                                                         PIP03080
   19 CONTINUE                                                          PIP03090
      READ(5,*) ALPHA,RE                                                PIP03100
      IF(ALPHA.LT.0.0D0) GO TO 180                                      PIP03110
C                                                                       PIP03120
CC*****************************************************************   PIP03130
C                                                                       PIP03140
CC                          PART III                                    PIP03150
CC                                                                      PIP03160
CC       COMPUTATION OF THE VORTICITY TRANSPORT EQUATION COEFFICIENTS   PIP03170
CC                                                                      PIP03180
CC       FOR P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES.            PIP03190
CC                                                                      PIP03200
CC       PRE-COMPUTE CONSTANTS INVOLVING ALPHA IN THE EQUATIONS.        PIP03210
C                                                                       PIP03220
      AL2 = ALPHA + ALPHA                                               PIP03230
      AL4 = AL2 + AL2                                                   PIP03240
      AL10 = 10.0D0 * ALPHA                                             PIP03250
      ALSQ = ALPHA * ALPHA                                              PIP03260
      AL4TH = ALSQ * ALSQ                                               PIP03270
      ALSQ2 = ALSQ + ALSQ                                               PIP03280
      ALSQ4 = ALSQ2 + ALSQ2                                             PIP03290
      ALSQ5 = ALSQ4 + ALSQ                                              PIP03300
      ALSQ10 = ALSQ5 + ALSQ5                                            PIP03310
      AL3RD2 = ALSQ * ALPHA * 2.0D0                                     PIP03320
C                                                                       PIP03330
CC       COMPUTE THE INCREMENTAL PIPE RADIUS AT INTERIOR MESH POINTS.   PIP03340
C                                                                       PIP03350
      DO 60 I = 1,MESH                                                  PIP03360
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      R = (DFLOAT(I) - 0.5D0) / DMESH                                   PIP03370
      R2 = R * R                                                        PIP03380
      R3 = R2 * R                                                       PIP03390
      R4 = R3 * R                                                       PIP03400
      RAD(I) = DCMPLX(R,0.0D0)                                          PIP03410
C                                                                       PIP03420
CC       VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)   PIP03430
CCC                                                                     PIP03440
CC       FIRST EQUATION  FIRST QUADRANT                                 PIP03450
C                                                                       PIP03460
      TEMPR = (1.0D0 + ALSQ * R2) / RE                                  PIP03470
      D2P1(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03480
      TEMPR = (( 3.0D0/R) + (R*ALSQ5)) / RE                             PIP03490
      D1P1(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03500
      TEMPR = ALSQ * ( 1.0D0 - ALSQ*R2) / RE                            PIP03510
      TEMPI = - AL2 * (1.0D0 - R2 + (ALSQ*R2) - (ALSQ*R4))              PIP03520
      D0P1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03530
      TEMPR = 1.0D0 + (ALSQ * R2)                                       PIP03540
      E0P1(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03550
C                                                                       PIP03560
CC       V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R) AND D3Q(R)          PIP03570
CCC                                                                     PIP03580
CC       FIRST EQUATION  SECOND QUADRANT                                PIP03590
C                                                                       PIP03600
      TEMPI = R / RE                                                    PIP03610
      D3Q1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03620
      TEMPI = 8.0D0 / RE                                                PIP03630
      D2Q1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03640
      TEMPR = AL2 *(R - R3)                                             PIP03650
      TEMPI = ((12.0D0/R) - (ALSQ*R)) / RE                              PIP03660
      D1Q1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03670
      TEMPR = AL2 * (3.0D0 - R2)                                        PIP03680
      TEMPI = - ALSQ5 / RE                                              PIP03690
      D0Q1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03700
      E1Q1(I) = DCMPLX(0.0D0,R)                                         PIP03710
      E0Q1(I) = DCMPLX(0.0D0,3.0D0)                                     PIP03720
C                                                                       PIP03730
CC       VORTICITY TRANSPORT EQN COEFFICIENTS FOR H(0)                  PIP03740
CCC                                                                     PIP03750
CC       FIRST EQUATION                                                 PIP03760
C                                                                       PIP03770
      TEMPR = AL2 * (2.0D0 - ALSQ + (ALSQ*R2))                          PIP03780
      TEMPI = AL4TH / RE                                                PIP03790
      H0A1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03800
      TEMPI = -ALSQ                                                     PIP03810
      H0B1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03820
C                                                                       PIP03830
C        VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)   PIP03840
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C                                                                       PIP03850
CC          SECOND EQUATION  THIRD QUADRANT                            PIP03860
C                                                                       PIP03870
      TEMPI = -4.0D0 / RE                                               PIP03880
      D2P2(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03890
      TEMPI = -12.0D0 / ( RE * R )                                      PIP03900
      D1P2(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03910
      TEMPR = -AL4 * (1.0D0 - R2)                                       PIP03920
      TEMPI = ALSQ4 / RE                                                PIP03930
      D0P2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03940
      E0P2(I) = DCMPLX(0.0D0,-2.0D0)                                    PIP03950
C                                                                       PIP03960
CC          V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R), D3Q(R) AND D4Q(R) PIP03970
CC                                                                      PIP03980
CC          SECOND EQUATION  FOURTH QUADRANT                          PIP03990
C                                                                       PIP04000
      TEMPR = -R2 / RE                                                  PIP04010
      D4Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP04020
      TEMPR = -(10.0D0 * R) / RE                                        PIP04030
      D3Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP04040
      TEMPR = (-19.0D0 +(ALSQ2 * R2))/ RE                               PIP04050
      TEMPI = AL2 * (R2 - R4)                                           PIP04060
      D2Q2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP04070
      TEMPR = ((3.0D0/R) + (ALSQ10*R)) / RE                             PIP04080
      TEMPI = AL10 * (R - R3)                                           PIP04090
      D1Q2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP04100
      TEMPR = (ALSQ4 -(AL4TH * R2))/ RE                                 PIP04110
      TEMPI = AL2 *((2.0D0 * (1.0D0 - R2))-(ALSQ * (R2 - R4)))          PIP04120
      D0Q2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP04130
      TEMPR = -R2                                                       PIP04140
      E2Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP04150
      TEMPR = -5.0D0 * R                                                PIP04160
      E1Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP04170
      TEMPR = -2.0D0 +(ALSQ * R2)                                       PIP04180
      E0Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP04190
C                                                                       PIP04200
CC          VORTICITY TRANSPORT EQN COEFFICIENTS FOR H(0)              PIP04210
CC                                                                      PIP04220
CC          SECOND EQUATION                                           PIP04230
C                                                                       PIP04240
      TEMPR = -AL4TH / RE                                               PIP04250
      TEMPI = -AL3RD2 * (1.0D0 -(ALPHA * R2))                           PIP04260
      H0A2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP04270
      H0B2(I) = DCMPLX(ALSQ,0.0D0)                                      PIP04280
   60 CONTINUE                                                          PIP04290
C                                                                       PIP04300
C***********************************************************************PIP04310
C                                                                       PIP04320
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C                                  PART IV                                      PIP04330
C                                                                               PIP04340
C        COMPUTATION OF THE A AND B MATRIX ELEMENTS.                            PIP04350
C                                                                               PIP04360
C        INITIALIZE THE A AND B MATRICES                                        PIP04370
C                                                                               PIP04380
      DO 198 I = 1,100                                                          PIP04390
      DO 198 J = 1,100                                                          PIP04400
      A(I,J) = ZERO                                                             PIP04410
      B(I,J) = ZERO                                                             PIP04420
  198 CONTINUE                                                                  PIP04430
      MH0 = 2 * MESH + 1                                                        PIP04440
      MQ0 = 2 * MESH + 2                                                        PIP04450
C                                                                               PIP04460
C        A AND B MATRIX ELEMENTS IN TERMS OF P(R)                               PIP04470
C                                                                               PIP04480
C        FIRST EQUATION  FIRST QUADRANT                                         PIP04490
C                                                                               PIP04500
      DO 200 J = 1,2                                                            PIP04510
      DO 200 K = 1,5                                                            PIP04520
      AODER = ZERO                                                              PIP04530
      IF (K.EQ.J) AODER = ONE                                                   PIP04540
      A(J,K) = AAXISP(2,J,K)*D2P1(J) + AAXISP(1,J,K)*D1P1(J) +                  PIP04550
     1         AODER*D0P1(J)                                                    PIP04560
      B(J,K) = AODER*E0P1(J)                                                    PIP04570
  200 CONTINUE                                                                  PIP04580
      MP = MESH - 2                                                             PIP04590
      DO 210 J = 3,MP                                                           PIP04600
      DO 210 K = 1,5                                                            PIP04610
      L = J - 3 + K                                                             PIP04620
      AODER = ZERO                                                              PIP04630
      IF(L.EQ.J) AODER = ONE                                                    PIP04640
      A(J,L) = A2DERP(K)*D2P1(J) + A1DERP(K)*D1P1(J) + AODER*D0P1(J)            PIP04650
      B(J,L) = AODER*E0P1(J)                                                    PIP04660
  210 CONTINUE                                                                  PIP04670
      MMP = MP - 3                                                              PIP04680
      DO 221 J = 1,2                                                            PIP04690
      L = J + MP                                                                PIP04700
      HOD1PA = ZERO                                                             PIP04710
      HOD2PA = ZERO                                                             PIP04720
      DO 220 K = 1,5                                                            PIP04730
      N = MMP + K                                                               PIP04740
      AODER = ZERO                                                              PIP04750
      IF(N.EQ.L) AODER = ONE                                                    PIP04760
      A(L,N) = AWALLP(2,J,K)*D2P1(L) + AWALLP(1,J,K)*D1P1(L) +                  PIP04770
     1         AODER*D0P1(L)                                                    PIP04780
      B(L,N) = AODER*E0P1(L)                                                    PIP04790
      HOD2PA = HOD2PA + AWALLP(2,J,K)                                           PIP04800
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      HOD1PA = HOD1PA + AWALLP(1,J,K)                                    PIP04810
  220 CONTINUE                                                           PIP04820
      HOD2PA = HOD2PA * D2P1(L)                                          PIP04830
      HOD1PA = HOD1PA * D1P1(L)                                          PIP04840
      HOA1(L) = HOA1(L) - (ONEIMG * (HOD2PA + HOD1PA))                   PIP04850
  221 CONTINUE                                                           PIP04860
C                                                                        PIP04870
CC       A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                        PIP04880
CCC                                                                      PIP04890
CC       FIRST EQUATION  SECOND QUADRANT                                 PIP04900
C                                                                        PIP04910
      DO 251 J = 1,3                                                     PIP04920
      QOD1QA = ZERO                                                      PIP04930
      QOD2QA = ZERO                                                      PIP04940
      QOD3QA = ZERO                                                      PIP04950
      QOD1QB = ZERO                                                      PIP04960
      DO 250 K = 1,6                                                     PIP04970
      KK = K + MESH                                                      PIP04980
      AODER = ZERO                                                       PIP04990
      IF(K.EQ.J) AODER = ONE                                             PIP05000
      A(J,KK) = AAXISQ(3,J,K)*D3Q1(J) + AAXISQ(2,J,K)*D2Q1(J) +          PIP05010
     1          AAXISQ(1,J,K)*D1Q1(J) + AODER*D0Q1(J)                    PIP05020
      B(J,KK) = AAXISQ(1,J,K)*E1Q1(J) + AODER*E0Q1(J)                    PIP05030
      AAXQ1 = AAXISQ(1,J,K)                                              PIP05040
      AAXQ2 = AAXISQ(2,J,K)                                              PIP05050
      IF(J.EQ.3) AAXQ1 = ZERO                                            PIP05060
      IF(J.EQ.3) AAXQ2 = ZERO                                            PIP05070
      QOD1QA = QOD1QA + AAXQ1                                            PIP05080
      QOD2QA = QOD2QA + AAXQ2                                            PIP05090
      QOD3QA = QOD3QA + AAXISQ(3,J,K)                                    PIP05100
      QOD1QB = QOD1QB + AAXQ1                                            PIP05110
  250 CONTINUE                                                           PIP05120
      QOD1QA = QOD1QA * D1Q1(J)                                          PIP05130
      QOD2QA = QOD2QA * D2Q1(J)                                          PIP05140
      QOD3QA = QOD3QA * D3Q1(J)                                          PIP05150
      QOD1QB = QOD1QB * E1Q1(J)                                          PIP05160
      A(J,MQQ) = -(QOD1QA + QOD2QA + QOD3QA)                             PIP05170
      B(J,MQQ) = -QOD1QB                                                 PIP05180
  251 CONTINUE                                                           PIP05190
      M = MESH - 3                                                       PIP05200
      DO 260 J = 4,M                                                     PIP05210
      DO 260 K = 1,7                                                     PIP05220
      L = J - 4 + K                                                      PIP05230
      LL = L + MESH                                                      PIP05240
      AODER = ZERO                                                       PIP05250
      IF(L.EQ.J) AODER = ONE                                             PIP05260
      A(J,LL) = A3DERQ(K)*D3Q1(J) + A2DERQ(K)*D2Q1(J) +                  PIP05270
     1          A1DERQ(K)*D1Q1(J) + AODER*D0Q1(J)                        PIP05280
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      B(J,LL) =  A1DERQ(K)*E1Q1(J) + A0DER*E0Q1(J)                       PIP05290
  260 CONTINUE                                                          PIP05300
      MM = M - 3                                                        PIP05310
      DO 266 J = 1,3                                                    PIP05320
      L = J + M                                                         PIP05330
      H0D3QA = ZERO                                                     PIP05340
      H0D2QA = ZERO                                                     PIP05350
      H0D1QA = ZERO                                                     PIP05360
      H0D1QB = ZERO                                                     PIP05370
      DO 265 K = 1,6                                                    PIP05380
      KK = 7 - K                                                        PIP05390
      N = MM + K                                                        PIP05400
      NN = N + MESH                                                     PIP05410
      A0DER = ZERO                                                      PIP05420
      IF(N.EQ.L) A0DER = ONE                                            PIP05430
      A(L,NN) =  AWALLQ(3,J,K)*D3Q1(L) + AWALLQ(2,J,K)*D2Q1(L) +        PIP05440
     1           AWALLQ(1,J,K)*D1Q1(L) + A0DER*D0Q1(L)                  PIP05450
      B(L,NN) =  AWALLQ(1,J,K)*E1Q1(L) + A0DER*E0Q1(L)                  PIP05460
      HTEMP = (1.0D0 - ((2.0D0 * DFLOAT(KK) - 1.0D0) / DMESH))          PIP05470
      BETA = DCMPLX(HTEMP,0.0D0)                                        PIP05480
      AWAQ1 = AWALLQ(1,J,K)                                             PIP05490
      AWAQ2 = AWALLQ(2,J,K)                                             PIP05500
      IF(J.EQ.1) AWAQ1 = ZERO                                           PIP05510
      IF(J.EQ.1) AWAQ2 = ZERO                                           PIP05520
      H0D1QA = H0D1QA + (AWAQ1 * BETA)                                  PIP05530
      H0D2QA = H0D2QA + (AWAQ2 * BETA)                                  PIP05540
      H0D3QA = H0D3QA + (AWALLQ(3,J,K) * BETA)                          PIP05550
      H0D1QB = H0D1QB + (AWAQ1 * BETA)                                  PIP05560
  265 CONTINUE                                                          PIP05570
      H0D1QA = H0D1QA + TWO                                             PIP05580
      IF(J.EQ.1) H0D1QA = ZERO                                          PIP05590
      H0D1QB = H0D1QB + TWO                                             PIP05600
      IF(J.EQ.1) H0D1QB = ZERO                                          PIP05610
      H0D1QA = H0D1QA * D1Q1(L)                                         PIP05620
      H0D2QA = H0D2QA * D2Q1(L)                                         PIP05630
      H0D3QA = H0D3QA * D3Q1(L)                                         PIP05640
      H0D1QB = H0D1QB * E1Q1(L)                                         PIP05650
      H0A1(L) = H0D1QA + H0D2QA + H0D3QA + H0A1(L)                      PIP05660
      H0B1(L) = H0D1QB + H0B1(L)                                        PIP05670
  266 CONTINUE                                                          PIP05680
C                                                                       PIP05690
CCC       A AND B MATRIX ELEMENTS IN TERMS OF H(0)                      PIP05700
CCC                                                                     PIP05710
CCC       FIRST EQUATION                                                PIP05720
C                                                                       PIP05730
      DO 267 I = 1,MESH                                                 PIP05740
      A(I,MH0) = H0A1(I)                                                PIP05750
      B(I,MH0) = H0B1(I)                                                PIP05760
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  267 CONTINUE                                                          PIP05770
C                                                                       PIP05780
CCCC        A AND B MATRIX ELEMENTS IN TERMS OF P(R)                    PIP05790
CCCC                                                                    PIP05800
CCCC        SECOND EQUATION  THIRD QUADRANT                             PIP05810
C                                                                       PIP05820
      DO 270 J = 1,2                                                    PIP05830
      JJ = J + MESH                                                     PIP05840
      DO 270 K = 1,5                                                    PIP05850
      AODER = ZERO                                                      PIP05860
      IF(K.EQ.J) AODER = ONE                                            PIP05870
      A(JJ,K) = AAXISP(2,J,K)*D2P2(J) + AAXISP(1,J,K)*D1P2(J) +         PIP05880
     1          AODER*DOP2(J)                                           PIP05890
      B(JJ,K) = AODER*EOP2(J)                                           PIP05900
  270 CONTINUE                                                          PIP05910
      DO 275 J = 3,MP                                                   PIP05920
      JJ = J + MESH                                                     PIP05930
      DO 275 K = 1,5                                                    PIP05940
      L = J - 3 + K                                                     PIP05950
      AODER = ZERO                                                      PIP05960
      IF(L.EQ.J) AODER = ONE                                            PIP05970
      A(JJ,L) = A2DERP(K)*D2P2(J) + A1DERP(K)*D1P2(J) + AODER*DOP2(J)   PIP05980
      B(JJ,L) = AODER*EOP2(J)                                           PIP05990
  275 CONTINUE                                                          PIP06000
      DO 281 J = 1,2                                                    PIP06010
      L = J + MP                                                        PIP06020
      LL = L + MESH                                                     PIP06030
      HOD1PA = ZERO                                                     PIP06040
      HOD2PA = ZERO                                                     PIP06050
      DO 280 K = 1,5                                                    PIP06060
      N = MMP + K                                                       PIP06070
      AODER = ZERO                                                      PIP06080
      IF (N.EQ.L) AODER = ONE                                           PIP06090
      A(LL,N) = AWALLP(2,J,K)*D2P2(L) + AWALLP(1,J,K)*D1P2(L) +         PIP06100
     1          AODER*DOP2(L)                                           PIP06110
      B(LL,N) = AODER*EOP2(L)                                           PIP06120
      HOD1PA = HOD1PA + AWALLP(1,J,K)                                   PIP06130
      HOD2PA = HOD2PA + AWALLP(2,J,K)                                   PIP06140
  280 CONTINUE                                                          PIP06150
      HOD1PA = HOD1PA * D1P2(L)                                         PIP06160
      HOD2PA = HOD2PA * D2P2(L)                                         PIP06170
      HOA2(L) = HOA2(L) - (ONEIMG * (HOD1PA + HOD2PA))                  PIP06180
  281 CONTINUE                                                          PIP06190
C                                                                       PIP06200
CCCC        A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                    PIP06210
CCCC                                                                    PIP06220
CCCC        SECOND EQUATION  FOURTH QUADRANT                            PIP06230
C                                                                       PIP06240
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      DO 286 J = 1,3                                                                    PIP06250
      JJ = J + MESH                                                                     PIP06260
      QQD1QA = ZERO                                                                     PIP06270
      QQD2QA = ZERO                                                                     PIP06280
      QQD3QA = ZERO                                                                     PIP06290
      QQD4QA = ZERO                                                                     PIP06300
      QQD1QB = ZERO                                                                     PIP06310
      QQD2QB = ZERO                                                                     PIP06320
      DO 285 K = 1,6                                                                    PIP06330
      KK = K + MESH                                                                     PIP06340
      AODER = ZERO                                                                      PIP06350
      IF(K.EQ.J) AODER = ONE                                                            PIP06360
      A(JJ,KK) = AAXISQ(4,J,K)*D4Q2(J) + AAXISQ(3,J,K)*D3Q2(J) +                        PIP06370
     1           AAXISQ(2,J,K)*D2Q2(J) + AAXISQ(1,J,K)*D1Q2(J) +                        PIP06380
     2           AODER*DOQ2(J)                                                         PIP06390
      B(JJ,KK) = AAXISQ(2,J,K)*E2Q2(J) + AAXISQ(1,J,K)*E1Q2(J) +                        PIP06400
     1           AODER*EOQ2(J)                                                         PIP06410
      AAXQ1 = AAXISQ(1,J,K)                                                             PIP06420
      AAXQ2 = AAXISQ(2,J,K)                                                             PIP06430
      IF(J.EQ.3) AAXQ1 = ZERO                                                           PIP06440
      IF(J.EQ.3) AAXQ2 = ZERO                                                           PIP06450
      QQD1QA = QQD1QA + AAXQ1                                                           PIP06460
      QQD2QA = QQD2QA + AAXQ2                                                           PIP06470
      QQD3QA = QQD3QA + AAXISQ(3,J,K)                                                   PIP06480
      QQD4QA = QQD4QA + AAXISQ(4,J,K)                                                   PIP06490
      QQD1QB = QQD1QB + AAXQ1                                                           PIP06500
      QQD2QB = QQD2QB + AAXQ2                                                           PIP06510
  285 CONTINUE                                                                          PIP06520
      QQD1QA = QQD1QA * D1Q2(J)                                                         PIP06530
      QQD2QA = QQD2QA * D2Q2(J)                                                         PIP06540
      QQD3QA = QQD3QA * D3Q2(J)                                                         PIP06550
      QQD4QA = QQD4QA * D4Q2(J)                                                         PIP06560
      QQD1QB = QQD1QB * E1Q2(J)                                                         PIP06570
      QQD2QB = QQD2QB * E2Q2(J)                                                         PIP06580
      A(JJ,MQQ) = -(QQD1QA + QQD2QA + QQD3QA + QQD4QA)                                  PIP06590
      B(JJ,MQQ) = -(QQD1QB + QQD2QB)                                                    PIP06600
  286 CONTINUE                                                                          PIP06610
      DO 290 J = 4,M                                                                    PIP06620
      JJ = J + MESH                                                                     PIP06630
      DO 290 K = 1,7                                                                    PIP06640
      L = J - 4 + K                                                                     PIP06650
      LL = L + MESH                                                                     PIP06660
      AODER = ZERO                                                                      PIP06670
      IF(L.EQ.J) AODER = ONE                                                            PIP06680
      A(JJ,LL) = A4DERQ(K)*D4Q2(J) + A3DERQ(K)*D3Q2(J) +                                PIP06690
     1           A2DERQ(K)*D2Q2(J) + A1DERQ(K)*D1Q2(J) + AODER*DOQ2(J)                  PIP06700
      B(JJ,LL) = A2DERQ(K)*E2Q2(J) + A1DERQ(K)*E1Q2(J) + AODER*EOQ2(J)                  PIP06710
  290 CONTINUE                                                                          PIP06720
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      DO 296 J = 1,3                                                                PIP06730
      L = J + M                                                                     PIP06740
      LL = L + MESH                                                                 PIP06750
      HOD1QA = ZERO                                                                 PIP06760
      HOD2QA = ZERO                                                                 PIP06770
      HOD3QA = ZERO                                                                 PIP06780
      HOD4QA = ZERO                                                                 PIP06790
      HOD1QB = ZERO                                                                 PIP06800
      HOD2QB = ZERO                                                                 PIP06810
      DO 295 K = 1,6                                                                PIP06820
      KK = 7 - K                                                                    PIP06830
      N = MM + K                                                                    PIP06840
      NN = N + MESH                                                                 PIP06850
      AODER = ZERO                                                                  PIP06860
      IF(N.EQ.L) AODER = ONE                                                        PIP06870
      A(LL,NN) = AWALLQ(4,J,K)*D4Q2(L) + AWALLQ(3,J,K)*D3Q2(L) +                    PIP06880
     1           AWALLQ(2,J,K)*D2Q2(L) + AWALLQ(1,J,K)*D1Q2(L) +                    PIP06890
     2           AODER*DOQ2(L)                                                      PIP06900
      B(LL,NN) = AWALLQ(2,J,K)*E2Q2(L) + AWALLQ(1,J,K)*E1Q2(L) +                    PIP06910
     1           AODER*E0Q2(L)                                                      PIP06920
      HTEMP = (1.0D0 - ((2.0D0 * DFLOAT(KK) - 1.0D0) / DMESH))                      PIP06930
      BETA = DCMPLX(HTEMP,0.0D0)                                                    PIP06940
      AWAQ1 = AWALLQ(1,J,K)                                                         PIP06950
      AWAQ2 = AWALLQ(2,J,K)                                                         PIP06960
      IF(J.EQ.1) AWAQ1 = ZERO                                                       PIP06970
      IF(J.EQ.1) AWAQ2 = ZERO                                                       PIP06980
      HOD1QA = HOD1QA + (AWAQ1 * BETA)                                              PIP06990
      HOD2QA = HOD2QA + (AWAQ2 * BETA)                                              PIP07000
      HOD3QA = HOD3QA + (AWALLQ(3,J,K) * BETA)                                      PIP07010
      HOD4QA = HOD4QA + (AWALLQ(4,J,K) * BETA)                                      PIP07020
      HOD1QB = HOD1QB + (AWAQ1 * BETA)                                              PIP07030
      HOD2QB = HOD2QB + (AWAQ2 * BETA)                                              PIP07040
  295 CONTINUE                                                                      PIP07050
      HOD1QA = HOD1QA + TWO                                                         PIP07060
      HOD1QB = HOD1QB + TWO                                                         PIP07070
      IF(J.EQ.1) HOD1QA = ZERO                                                      PIP07080
      IF(J.EQ.1) HOD1QB = ZERO                                                      PIP07090
      HOD1QA = HOD1QA * D1Q2(L)                                                     PIP07100
      HOD2QA = HOD2QA * D2Q2(L)                                                     PIP07110
      HOD3QA = HOD3QA * D3Q2(L)                                                     PIP07120
      HOD4QA = HOD4QA * D4Q2(L)                                                     PIP07130
      HOD1QB = HOD1QB * E1Q2(L)                                                     PIP07140
      HOD2QB = HOD2QB * E2Q2(L)                                                     PIP07150
      HOA2(L) = HOA2(L) + HOD1QA + HOD2QA + HOD3QA + HOD4QA                         PIP07160
      HOB2(L) = HOB2(L) + HOD1QB + HOD2QB                                           PIP07170
  296 CONTINUE                                                                      PIP07180
C                                                                                   PIP07190
C        A AND B MATRIX ELEMENTS IN TERMS OF H(0)                                   PIP07200
```

```
C                                                                       PIP07210
CC        SECOND EQUATION                                               PIP07220
C                                                                       PIP07230
      DO 297 I = 1,MESH                                                 PIP07240
      II = I + MESH                                                     PIP07250
      A(II,MH0)  = H0A2(I)                                              PIP07260
      B(II,MH0)  = H0B2(I)                                              PIP07270
  297 CONTINUE                                                          PIP07280
C                                                                       PIP07290
CCCC      A AND B MATRIX ELEMENTS FOR THE TWO SPECIAL BOUNDARY          PIP07300
CCCCC     CONDITION EQUATIONS INVOLVING GAMMA.                          PIP07310
CCCCC                                                                   PIP07320
CCCC      FIRST EQUATION COEFFICIENTS                                   PIP07330
C                                                                       PIP07340
      TEMPR = 4.0D0 / RE                                                PIP07350
      D2P01 = DCMPLX(TEMPR,0.0D0)                                       PIP07360
      TEMPI = 20.0D0 / RE                                               PIP07370
      D2Q01 = DCMPLX(0.0D0,TEMPI)                                       PIP07380
      TEMPR = ALSQ / RE                                                 PIP07390
      TEMPI = - AL2                                                     PIP07400
      D0P01 = DCMPLX(TEMPR,TEMPI)                                       PIP07410
      TEMPR = 6.0D0 * ALPHA                                             PIP07420
      TEMPI = - ALSQ5 / RE                                              PIP07430
      D0Q01 = DCMPLX(TEMPR,TEMPI)                                       PIP07440
      TEMPR = AL4 - AL3RD2                                              PIP07450
      TEMPI = AL4TH / RE                                                PIP07460
      D0H01 = DCMPLX(TEMPR,TEMPI)                                       PIP07470
      E0P01 = ONE                                                       PIP07480
      E0Q01 = DCMPLX(0.0D0,3.0D0)                                       PIP07490
      TEMPI = - ALSQ                                                    PIP07500
      E0H01 = DCMPLX(0.0D0,TEMPI)                                       PIP07510
C                                                                       PIP07520
CCC       SECOND EQUATION COEFFICIENTS                                  PIP07530
C                                                                       PIP07540
      TEMPI = - 16.0D0 / RE                                             PIP07550
      D2P02 = DCMPLX(0.0D0,TEMPI)                                       PIP07560
      D2Q02 = DCMPLX(TEMPI,0.0D0)                                       PIP07570
      TEMPR = -AL4                                                      PIP07580
      TEMPI = ALSQ4 / RE                                                PIP07590
      D0P02 = DCMPLX(TEMPR,TEMPI)                                       PIP07600
      D0Q02 = DCMPLX(TEMPI,AL4)                                         PIP07610
      TEMPR = - AL4TH / RE                                              PIP07620
      TEMPI = - AL3RD2                                                  PIP07630
      D0H02 = DCMPLX(TEMPR,TEMPI)                                       PIP07640
      E0P02 = DCMPLX(0.0D0,-2.0D0)                                      PIP07650
      E0Q02 = DCMPLX(-2.0D0,0.0D0)                                      PIP07660
      E0H02 = DCMPLX(ALSQ,0.0D0)                                        PIP07670
```

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C                                                                       PIP07690
CCC        A AND B MATRIX ELEMENTS RESULTING FROM THE TWO EQUATIONS     PIP07700
C                                                                       PIP07710
      Q0TEM1 = ZERO                                                     PIP07720
      Q0TEM2 = ZERO                                                     PIP07730
      DO 300 I = 1,6                                                    PIP07740
      A(MH0,I) = D2P01 * AI5D2P(I) + D0P01 * AI5P(I)                    PIP07750
      B(MH0,I) = E0P01 * AI5P(I)                                        PIP07760
      A(MQ0,I) = D2P02 * AI5D2P(I) + D0P02 * AI5P(I)                    PIP07770
      B(MQ0,I) = E0P02 * AI5P(I)                                        PIP07780
      II = I + MESH                                                     PIP07790
      A(MH0,II) = D2Q01 * AI6D2Q(I)                                     PIP07800
      Q0TEM1 = Q0TEM1 + AI5D2Q(I)                                       PIP07810
      A(MQ0,II) = D2Q02 * AI6D2Q(I)                                     PIP07820
      Q0TEM2 = Q0TEM2 + AI5D2Q(I)                                       PIP07830
  300 CONTINUE                                                          PIP07840
      A(MH0,MH0) = D0H01                                                PIP07850
      B(MH0,MH0) = E0H01                                                PIP07860
      A(MQ0,MH0) = D0H02                                                PIP07870
      B(MQ0,MH0) = E0H02                                                PIP07880
      A(MH0,MQ0) = D0Q01 - ( Q0TEM1 * D2Q01)                            PIP07890
      B(MH0,MQ0) = E0Q01                                                PIP07900
      A(MQ0,MQ0) = D0Q02 - ( Q0TEM2 * D2Q02)                            PIP07910
      B(MQ0,MQ0) = E0Q02                                                PIP07920
C                                                                       PIP07930
C**********************************************************************PIP07940
CCC                                                                     PIP07950
CCC                          PART V                                     PIP07960
CCC                                                                     PIP07970
CCC        COMPUTATION OF EIGENVALUES AND RESPECTIVE EIGENVECTORS       PIP07980
CCC                                                                     PIP07990
CCC        USING THE SUBROUTINE EIGZC WITH THE VORTICITY TRANSPORT      PIP08000
CCC                                                                     PIP08010
CCC        EQUATIONS IN THE FORM  A * X = GAMMA * B * X.                PIP08020
C                                                                       PIP08030
      DO 140 I = 1,100                                                  PIP08040
      EIGVAL(I) = ZERO                                                  PIP08050
      EIGA(I) = ZERO                                                    PIP08060
      EIGB(I) = ZERO                                                    PIP08070
      DO 140 J = 1,100                                                  PIP08080
      EIGVEC(I,J) = ZERO                                                PIP08090
      AA(I,J) = A(I,J)                                                  PIP08100
      BB(I,J) = B(I,J)                                                  PIP08110
  140 CONTINUE                                                          PIP08120
      WRITE(6,145)                                                      PIP08130
  145 FORMAT(1X,//,1X,'**************************',                     PIP08140
     1   '**************************************',                     PIP08150
     2   '**************************************',//)                  PIP08160
```

```
      MESH2 = (MESH * 2) + 2                                            PIP08170
      IJOB = 1                                                          PIP08180
      CALL EIGZC(AA,100,BB,100,MESH2,IJOB,EIGA,EIGB,                    PIP08190
     1EIGVEC,100,WK,INEER,IER)                                          PIP08200
      IF(INEER.EQ.0) WRITE(6,150)                                       PIP08210
  150 FORMAT(1X,////,10X,'ALL EIGENVALUES CONVERGED WITHIN 30 ',        PIP08220
     1     'ITERATIONS',//)                                           PIP08230
      IF(INEER.NE.0) WRITE(6,155) INEER                                 PIP08240
  155 FORMAT(1X,////,10X,'FAILED TO CONVERGE ON THE',I4,1X,             PIP08250
     1     'TH EIGENVALUE',//)                                        PIP08260
      DO 165 I = 1,MESH2                                                PIP08270
      EIGVAL(I) = EIGA(I)/EIGB(I)                                       PIP08280
      GAMR(I) = DREAL(EIGVAL(I))                                        PIP08290
  165 CONTINUE                                                          PIP08300
      NGAM = 1                                                          PIP08310
      GAMMAX = GAMR(1)                                                  PIP08320
      DO 170 I = 2,MESH2                                                PIP08330
      IF(GAMR(I).LT.GAMMAX) GO TO 169                                   PIP08340
      GAMMAX = GAMR(I)                                                  PIP08350
      NGAM = I                                                          PIP08360
  169 CONTINUE                                                          PIP08370
  170 CONTINUE                                                          PIP08380
      WRITE(6,175) NGAM,GAMMAX,MESH,ALPHA,RE                            PIP08390
  175 FORMAT(1X,/,1X,'LEAST STABLE EIGENVALUE NR',I3,1X,'=',D27.18,4X,  PIP08400
     1 'MESH =',I3,4X,'ALPHA =',D10.3,4X,'REYNOLDS NR =',D17.10,/)      PIP08410
C                                                                       PIP08420
CC*********************************************************************PIP08430
C                                                                       PIP08440
C                              PART VI                                  PIP08450
C                                                                       PIP08460
C        VERIFY THAT THE LEAST STABLE EIGENVALUE AND CORRESPONDING      PIP08470
C                                                                       PIP08480
C        EIGENVECTOR SATISFY THE GOVERNING PARTIAL DIFFERENTIAL         PIP08490
C                                                                       PIP08500
C        EQUATIONS.  RESIDUAL ERROR IS OUTPUT AS EPSILON.               PIP08510
C                                                                       PIP08520
      DO 351 I = 1,MESH2                                                PIP08530
      SUMA = ZERO                                                       PIP08540
      SUMB = ZERO                                                       PIP08550
      DO 350 J = 1,MESH2                                                PIP08560
      SUMA = SUMA + ( A(I,J) * EIGVEC(J,NGAM) )                         PIP08570
      SUMB = SUMB + ( B(I,J) * EIGVEC(J,NGAM) )                         PIP08580
  350 CONTINUE                                                          PIP08590
      ERROR(I) = SUMA - ( SUMB * EIGVAL(NGAM) )                         PIP08600
  351 CONTINUE                                                          PIP08610
      WRITE(6,375)                                                      PIP08620
  375 FORMAT(1X,//,15X,'EPSILON = A * X  -  GAMMA * B * X',/)           PIP08630
      WRITE(6,380) (ERROR(I),I = 1,MESH2)                               PIP08640
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  380 FORMAT(1X,2D30.18)                                                        PIP08650
C                                                                               PIP08660
CC*****************************************************************          PIP08670
C                                                                               PIP08680
C                                PART VII                                       PIP08690
C                                                                               PIP08700
C       COMPUTATION OF AXIAL PERTURBATION VELOCITY  U(R).                       PIP08710
C                                                                               PIP08720
C       OUTPUT THE EIGENVECTOR CORRESPONDING TO THE LEASE STABLE                PIP08730
C                                                                               PIP08740
C       EIGENVALUE,  GAMMA*.                                                    PIP08750
C                                                                               PIP08760
      DO 400 I = 1,MESH                                                         PIP08770
      II = I + MESH                                                             PIP08780
      P(I) = EIGVEC(I,NGAM)                                                     PIP08790
      Q(I) = EIGVEC(II,NGAM)                                                    PIP08800
  400 CONTINUE                                                                  PIP08810
      H0 = EIGVEC(MH0,NGAM)                                                     PIP08820
      Q0 = EIGVEC(MQ0,NGAM)                                                     PIP08830
      WRITE(6,401) EIGVAL(NGAM)                                                 PIP08840
  401 FORMAT(1X,///,5X,'EIGENVECTOR CORRESPONDING TO',                          PIP08850
     1       ' THE LEAST STABLE EIGENVALUE, GAMMA* =',2D27.18,/)                PIP08860
      DO 402 I = 1,MESH                                                         PIP08870
      II = I + MESH                                                             PIP08880
      WRITE(6,404) I,P(I),II,Q(I)                                               PIP08890
  404 FORMAT(2X,I2,2D28.18,2X,I2,2D23.18)                                       PIP08900
  402 CONTINUE                                                                  PIP08910
      WRITE(6,403) H0,Q0                                                        PIP08920
  403 FORMAT(1X,/,2X,'H0',2D28.18,2X,'Q0',2D28.18,//)                           PIP08930
C                                                                               PIP08940
CC      COMPUTATION OF DQ(R)                                                    PIP08950
C                                                                               PIP08960
      DO 411 J = 1,3                                                            PIP08970
      DQ(J) = ZERO                                                              PIP08980
      QOD1QA = ZERO                                                             PIP08990
      DO 410 K = 1,6                                                            PIP09000
      AAXQ1 = AAXISQ(1,J,K)                                                     PIP09010
      IF(J.EQ.3) AAXQ1 = ZERO                                                   PIP09020
      DQ(J) = DQ(J) + Q(K) * AAXISQ(1,J,K)                                      PIP09030
      QOD1QA = QOD1QA + AAXQ1                                                   PIP09040
  410 CONTINUE                                                                  PIP09050
      DQ(J) = DQ(J) - (QOD1QA * Q0)                                             PIP09060
  411 CONTINUE                                                                  PIP09070
C                                                                               PIP09080
      DO 420 J = 4,M                                                            PIP09090
      DQ(J) = ZERO                                                              PIP09100
      DO 420 K = 1,7                                                            PIP09110
      I = J - 4 + K                                                             PIP09120
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      DQ(J) = DQ(J) +(Q(L) * A1DERQ(K))                                  PIP09130
  420 CONTINUE                                                           PIP09140
C                                                                        PIP09150
      DO 431 J = 1,3                                                     PIP09160
      L = J + M                                                          PIP09170
      DQ(L) = ZERO                                                       PIP09180
      HOD1QA = ZERO                                                      PIP09190
      DO 430 K = 1,6                                                     PIP09200
      AWAQ1 = AWALLQ(1,J,K)                                              PIP09210
      IF(J.EQ.1) AWAQ1 = ZERO                                            PIP09220
      KK = 7 - K                                                         PIP09230
      N = MM + K                                                         PIP09240
      HTEMP =  1.0D0 - ((2.0D0 * DFLOAT(KK) - 1.0D0) / DMESH)            PIP09250
      BETA = DCMPLX(HTEMP,0.0D0)                                         PIP09260
      HOD1QA = HOD1QA + (AWAQ1 * BETA)                                   PIP09270
      DQ(L) = DQ(L) +(Q(N) * AWALLQ(1,J,K))                              PIP09280
  430 CONTINUE                                                           PIP09290
      HOD1QA = HOD1QA + TWO                                              PIP09300
      IF(J.EQ.1) HOD1QA = ZERO                                           PIP09310
      DQ(L) = DQ(L) + (HOD1QA * HO)                                      PIP09320
  431 CONTINUE                                                           PIP09330
C                                                                        PIP09340
      DO 440 I = 1,MESH                                                  PIP09350
      U(I) = RAD(I) * ((THREE*Q(I)) + (RAD(I)*DQ(I)) - (ONEIMG*P(I)))    PIP09360
  440 CONTINUE                                                           PIP09370
C                                                                        PIP09380
CC       REARRANGE DATA FOR PLOT ROUTINE                                 PIP09390
C                                                                        PIP09400
      M2 = MESH + 2                                                      PIP09410
      DO 450 I = 1,MESH                                                  PIP09420
      II = M2 - I                                                        PIP09430
      III = II - 1                                                       PIP09440
      U(II) = U(III)                                                     PIP09450
      RAD(II) = RAD(III)                                                 PIP09460
  450 CONTINUE                                                           PIP09470
      RAD(1) = ZERO                                                      PIP09480
      RAD(M2) = ONE                                                      PIP09490
      U(1) = ZERO                                                        PIP09500
      U(M2) = ZERO                                                       PIP09510
      DO 460 I = 1,M2                                                    PIP09520
      RADR(I) = DREAL(RAD(I))                                            PIP09530
  460 CONTINUE                                                           PIP09540
C                                                                        PIP09550
CC       COMPUTE THE NORMALIZED PERTURBATION VELOCITY AND                PIP09560
CCC                                                                      PIP09570
CC       NORMALIZED AMPLITUDE AND OUTPUT THE RESULTS.                    PIP09580
C                                                                        PIP09590
      DO 465 I = 1,M2                                                    PIP09600
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      UAMP(I) = CDABS(U(I))                                                  PIP09610
  465 CONTINUE                                                               PIP09620
      AMPMAX = UAMP(1)                                                       PIP09630
      IMAX = 1                                                               PIP09640
      DO 470 J = 2,M2                                                        PIP09650
      AMTEMP = UAMP(J)                                                       PIP09660
      IF(AMTEMP.LT.AMPMAX) GO TO 471                                         PIP09670
      AMPMAX = UAMP(J)                                                       PIP09680
      IMAX = J                                                               PIP09690
  471 CONTINUE                                                               PIP09700
  470 CONTINUE                                                               PIP09710
      WRITE(6,472)                                                           PIP09720
  472 FORMAT(1X,///,23X,'NORMALIZED PERTURBATION VELOCITY',                  PIP09730
     1 24X,'NORMALIZED AMPLITUDE',/)                                         PIP09740
      DO 475 K = 1,M2                                                        PIP09750
      UAMNOR(K) = UAMP(K) / AMPMAX                                           PIP09760
      AMPNOR(K) = UAMNOR(K)                                                  PIP09770
      UNORM(K) = U(K) / U(IMAX)                                              PIP09780
      URDP(K) = DREAL(UNORM(K))                                              PIP09790
      UR(K) = URDP(K)                                                        PIP09800
      UIDP(K) = DIMAG(UNORM(K))                                              PIP09810
      UI(K) = UIDP(K)                                                        PIP09820
      KK = K - 1                                                             PIP09830
      WRITE(6,466) KK,URDP(K),UIDP(K),UAMNOR(K)                              PIP09840
  466 FORMAT(1X,I3,1X,2D30.18,5X,D30.18)                                     PIP09850
  475 CONTINUE                                                               PIP09860
C                                                                            PIP09870
CC*****************************************************************        PIP09880
CCC                                                                          PIP09890
CCC                          PART VIII                                       PIP09900
CCC                                                                          PIP09910
CCC       PLOT THE NORMALIZED PERTURBATION VELOCITY VS PIPE RADIUS.          PIP09920
C                                                                            PIP09930
      IPLOT = IPLOT + 1                                                      PIP09940
      RMESH = FLOAT(MESH)                                                    PIP09950
      ALPLOT = ALPHA                                                         PIP09960
      REPLOT = RE                                                            PIP09970
      GAMPLT = GAMMAX                                                        PIP09980
      CALL PLOTG(RADR,UR,M2,1,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)        PIP09990
      CALL PLOTG(RADR,UI,M2,2,1,5,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)      PIP10000
C     CALL PLOTG(RADR,AMPNOR,M2,3,1,2,' ',1,' ',1,0.,1.,-1.,1.,5.,5.)        PIP10010
      CALL PLOTG(X1,Y1,2,3,1,0,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)         PIP10020
      XO = 0.15                                                              PIP10030
      YO = 5.6                                                               PIP10040
      IF(IPLOT.EQ.4) IPLOT = 1                                               PIP10050
      IF(IPLOT.EQ.2) YO = 12.1                                               PIP10060
      IF(IPLOT.EQ.3) YO = 18.6                                               PIP10070
      CALL TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                          PIP10080
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      GO TO 19                                                          PIP10090
  180 CONTINUE                                                          PIP10100
      CALL PLOT(0.0,0.0,999)                                            PIP10110
      STOP                                                              PIP10120
      END                                                               PIP10130
C                                                                       PIP10140
C.....................................................................  PIP10150
C                                                                       PIP10160
C        SUBROUTINE TITLE1                                              PIP10170
C                                                                       PIP10180
C        SUBROUTINE TO WRITE TITLE ON PLOTS                             PIP10190
C                                                                       PIP10200
      SUBROUTINE TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)               PIP10210
      RN = 1.0                                                          PIP10220
      YHT = 0.14                                                        PIP10230
      CALL NEWPEN(2)                                                    PIP10240
      CALL SYMBOL(XO,YO,YHT,                                            PIP10250
     1'NORMALIZED PERTURBATION VELOCITY VS RADIUS',0.0,42)              PIP10260
      YHT = 0.08                                                        PIP10270
      DELY = 0.20                                                       PIP10280
      CALL NEWPEN(1)                                                    PIP10290
      YO = YO - 0.8                                                     PIP10300
      XO = XO + 2.5                                                     PIP10310
      XN = XO + .75                                                     PIP10320
      CALL SYMBOL(XO,YO,YHT,'N      = ',0.0,9)                          PIP10330
      CALL NUMBER(XN,YO,YHT,RN,0.0,-1)                                  PIP10340
      YO = YO - DELY                                                    PIP10350
      CALL SYMBOL(XO,YO,YHT,'MESH   = ',0.0,9)                          PIP10360
      CALL NUMBER(XN,YO,YHT,RMESH,0.0,-1)                               PIP10370
      YO = YO - DELY                                                    PIP10380
      CALL SYMBOL(XO,YO,YHT,'ALPHA  = ',0.0,9)                          PIP10390
      CALL NUMBER(XN,YO,YHT,ALPLOT,0.0,2)                               PIP10400
      YO = YO - DELY                                                    PIP10410
      CALL SYMBOL(XO,YO,YHT,'REY NR = ',0.0,9)                          PIP10420
      CALL NUMBER(XN,YO,YHT,REPLOT,0.0,1)                               PIP10430
      YO = YO - DELY                                                    PIP10440
      CALL SYMBOL(XO,YO,YHT,'GAMMA* = ',0.0,9)                          PIP10450
      CALL NUMBER(XN,YO,YHT,GAMPLT,0.0,4)                               PIP10460
      YO = YO - 3.4                                                     PIP10470
      CALL SYMBOL(XO,YO,YHT,'OCTAGON = U(REAL)',0.0,17)                 PIP10480
      YO = YO - DELY                                                    PIP10490
      CALL SYMBOL(XO,YO,YHT,'DIAMOND = U(IMAG)',0.0,17)                 PIP10500
      YO = YO - DELY                                                    PIP10510
C     CALL SYMBOL(XO,YO,YHT,'TRIANGLE= U AMPL ',0.0,17)                 PIP10520
CCC                                                                     PIP10530
CCC      WRITE THE AXIS LABLES                                          PIP10540
C                                                                       PIP10550
      XO = XO - 2.55                                                    PIP10560
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      YO = YO + 0.3                                                          PIP10570
      YHT = 0.12                                                             PIP10580
      CALL SYMBOL(XO,YO,YHT,'NORMALIZED PERTURBATION VELOCITY',90.,32)       PIP10590
      XO = XO + 2.8                                                          PIP10600
      YO = YO - 1.0                                                          PIP10610
      CALL SYMBOL(XO,YO,YHT,'PIPE RADIUS',0.0,11)                            PIP10620
      RETURN                                                                 PIP10630
      END                                                                    PIP10640
/*                                                                           PIP10650
//GO.SYSIN DD *                                                            PIP10660
2 1 5                                                                        PIP10670
     -0.75414446417998842 7D+00                                              PIP10680
      0.37221235445036506 0D+00                                              PIP10690
      0.61900532859680124 1D+00                                              PIP10700
     -0.28537596210775519 1D+00                                              PIP10710
      0.48302743240576178 3D-01                                              PIP10720
2 2 5                                                                        PIP10730
      0.12837477797513325 9D+01                                              PIP10740
     -0.25705545687783668 5D+01                                              PIP10750
      0.13880994671403148 5D+01                                              PIP10760
     -0.10479573712255696 4D+00                                              PIP10770
      0.35030590092751806 6D-02                                              PIP10780
1 1 4                                                                        PIP10790
     -0.10265151515151513 8D+01                                              PIP10800
      0.13977272727272712 1D+01                                              PIP10810
     -0.44318181818181813 5D+00                                              PIP10820
      0.71969696969694740 1D-01                                              PIP10830
1 2 4                                                                        PIP10840
     -0.60227272727267264 05D+00                                             PIP10850
      0.34090909090907173 3D-01                                              PIP10860
      0.64772727272732733 7D+00                                              PIP10870
     -0.79545454545444700 1D-01                                              PIP10880
9 0 0                                                                        PIP10890
      0.13769010879218464 0D+01                                              PIP10900
     -0.61195603907637506 6D+00                                              PIP10910
      0.33045626110124208 5D+00                                              PIP10920
     -0.11240008880994618 9D+00                                              PIP10930
      0.16998778863232564 4D-01                                              PIP10940
     -0.44265097690941326 3D+01                                              PIP10950
      0.78006710084862640 5D+01                                              PIP10960
     -0.48423623445825754 2D+01                                              PIP10970
      0.17388987566607383 4D+01                                              PIP10980
     -0.27069765147029635 5D+00                                              PIP10990
2 1 5                                                                        PIP11000
     -0.54166666666666580 8D+01                                              PIP11010
      0.24444444444444417 6D+01                                              PIP11020
     -0.49999999999994837 D+00                                               PIP11030
      0.95238095238094885 9D-01                                              PIP11040
```

```
      -0.9259259259258898790-02                                         PIP11050
2 2 5                                                                   PIP11060
       0.1749999999999992450+01                                         PIP11070
      -0.2777777777777775900+01                                         PIP11080
       0.1499999999999991780+01                                         PIP11090
      -0.1428571428571396850+00                                         PIP11100
       0.9259259259258759140-02                                         PIP11110
1 1 4                                                                   PIP11120
       0.1666666666666658650+00                                         PIP11130
       0.1000000000000000022D+01                                        PIP11140
      -0.2999999999999998920+00                                         PIP11150
       0.4761904761904763380-01                                         PIP11160
1 2 4                                                                   PIP11170
      -0.9999999999999973350+00                                         PIP11180
       0.1666666666666642980+00                                         PIP11190
       0.6000000000000011990+00                                         PIP11200
      -0.7142857142857128580-01                                         PIP11210
9 0 0                                                                   PIP11220
4 1 6                                                                   PIP11230
       0.1150000000000007140+03                                         PIP11240
      -0.3844444444444502550+02                                         PIP11250
       0.1640000000000038580+02                                         PIP11260
      -0.5224489795918497940+01                                         PIP11270
       0.1123456790123495090+01                                         PIP11280
      -0.1157024793388475580+00                                         PIP11290
4 2 6                                                                   PIP11300
      -0.2600000000000080290+02                                         PIP11310
       0.1766666666666701600+02                                         PIP11320
      -0.1120000000000023020+02                                         PIP11330
       0.3959183673469510720+01                                         PIP11340
      -0.6666666666667004910+00                                         PIP11350
       0.5785123966942518070-01                                         PIP11360
4 3 6                                                                   PIP11370
       0.1000000000000852650+01                                         PIP11380
      -0.6222222222222804080+01                                         PIP11390
       0.9200000000000457590+01                                         PIP11400
      -0.6448979591836899050+01                                         PIP11410
       0.1987654320987697790+01                                         PIP11420
      -0.1652892561983525520+00                                         PIP11430
3 1 5                                                                   PIP11440
      -0.1749999999999997510+02                                         PIP11450
       0.1111111111111076030+00                                         PIP11460
       0.1200000000000013720+01                                         PIP11470
      -0.3469387755102073950+00                                         PIP11480
       0.4320987654321039110-01                                         PIP11490
3 2 5                                                                   PIP11500
       0.1349999999999996800+02                                         PIP11510
      -0.3888888888888882130+01                                         PIP11520
```

```
      -0.14210854715202003 7D-13                                        PIP11530
       0.55102040816326658 4D+00                                        PIP11540
      -0.67901234567901785 9D-01                                        PIP11550
3 3 5                                                                   PIP11560
      -0.15500000000000113 7D+02                                        PIP11570
       0.98888888888891060 3D+01                                        PIP11580
      -0.60000000000000532 9D+01                                        PIP11590
       0.14489795918367498 5D+01                                        PIP11600
       0.67901234567899093 6D-01                                        PIP11610
2 1 4                                                                   PIP11620
      -0.46666666666666658 5D+01                                        PIP11630
       0.21111111111111102 7D+01                                        PIP11640
      -0.31999999999999993 3D+00                                        PIP11650
       0.34013605442176852 1D-01                                        PIP11660
2 2 4                                                                   PIP11670
       0.99999999999978684 4D+00                                        PIP11680
      -0.24444444444444493 0D+01                                        PIP11690
       0.13200000000000011 7D+01                                        PIP11700
      -0.81632653061224511 0D-01                                        PIP11710
1 1 3                                                                   PIP11720
       0.24999999999999889 3D+01                                        PIP11730
       0.22222222222222293 3D+00                                        PIP11740
      -0.19999999999999978 3D-01                                        PIP11750
1 2 3                                                                   PIP11760
      -0.45000000000000035 5D+01                                        PIP11770
       0.13333333333333304 4D+01                                        PIP11780
       0.18000000000000007 3D+00                                        PIP11790
9 0 0                                                                   PIP11800
       0.21656250000000071 1D+02                                        PIP11810
      -0.40104166666667335 8D+01                                        PIP11820
       0.17325000000000434 5D+01                                        PIP11830
      -0.63137755102042347 8D+00                                        PIP11840
       0.14853395061728838 5D+00                                        PIP11850
      -0.16270661157025321 2D-01                                        PIP11860
4 1 6                                                                   PIP11870
       0.11500000000000071 4D+03                                        PIP11880
      -0.38444444444445025 5D+02                                        PIP11890
       0.16400000000000385 8D+02                                        PIP11900
      -0.52244897959184979 4D+01                                        PIP11910
       0.11234567901234950 9D+01                                        PIP11920
      -0.11570247933884755 8D+00                                        PIP11930
4 2 6                                                                   PIP11940
      -0.26000000000000802 9D+02                                        PIP11950
       0.17666666666667016 0D+02                                        PIP11960
      -0.11200000000000230 2D+02                                        PIP11970
       0.39591836734695107 2D+01                                        PIP11980
      -0.66666666666670049 1D+00                                        PIP11990
       0.57851239669425180 7D-01                                        PIP12000
```

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4 3 6                                                                   PIP12010
      0.10000000000085265D+01                                           PIP12020
     -0.62222222222804080D+01                                           PIP12030
      0.92000000000045759D+01                                           PIP12040
     -0.64489795918368990D+01                                           PIP12050
      0.19876543209876977 9D+01                                         PIP12060
     -0.16528925619835252 0D+00                                         PIP12070
3 1 5                                                                   PIP12080
     -0.17499999999999751D+02                                           PIP12090
      0.11111111111107608D+00                                           PIP12100
      0.12000000000001372D+01                                           PIP12110
     -0.34693877551020739 5D+00                                         PIP12120
      0.43209876543210391 1D-01                                         PIP12130
3 2 5                                                                   PIP12140
      0.13499999999999680D+02                                           PIP12150
     -0.38888888888888213D+01                                           PIP12160
     -0.14210854715202003 7D-13                                         PIP12170
      0.55102040816326658 4D+00                                         PIP12180
     -0.67901234567901785 9D-01                                         PIP12190
3 3 5                                                                   PIP12200
     -0.15500000000000113 7D+02                                         PIP12210
      0.98888888888891060D+01                                           PIP12220
     -0.60000000000000532 9D+01                                         PIP12230
      0.14489795918367498 5D+01                                         PIP12240
      0.67901234567899093 5D-01                                         PIP12250
2 1 4                                                                   PIP12260
     -0.46666666666666585D+01                                           PIP12270
      0.21111111111111102 7D+01                                         PIP12280
     -0.31999999999999993D+00                                           PIP12290
      0.34013605442176852 1D-01                                         PIP12300
2 2 4                                                                   PIP12310
      0.99999999999978684D+00                                           PIP12320
     -0.24444444444444493 0D+01                                         PIP12330
      0.13200000000000011 7D+01                                         PIP12340
     -0.81632653061224511 0D-01                                         PIP12350
1 1 3                                                                   PIP12360
      0.24999999999999889D+01                                           PIP12370
      0.22222222222222293D+00                                           PIP12380
     -0.19999999999999973D-01                                           PIP12390
1 2 3                                                                   PIP12400
     -0.45000000000000035 5D+01                                         PIP12410
      0.13333333333333304D+01                                           PIP12420
      0.18000000000000007D+00                                           PIP12430
9 0 0                                                                   PIP12440
0.0D0     1150.0D0                                                      PIP12450
0.5D0     1150.0D0                                                      PIP12460
1.0D0     1150.0D0                                                      PIP12470
2.0D0     1150.0D0                                                      PIP12480
```

```
4.0D0     1150.0D0                                                  PIP12490
8.0D0     1150.0D0                                                  PIP12500
16.0D0    1150.0D0                                                  PIP12510
32.0D0    1150.0D0                                                  PIP12520
0.0D0     2300.0D0                                                  PIP12530
0.5D0     2300.0D0                                                  PIP12540
1.0D0     2300.0D0                                                  PIP12550
2.0D0     2300.0D0                                                  PIP12560
4.0D0     2300.0D0                                                  PIP12570
8.0D0     2300.0D0                                                  PIP12580
16.0D0    2300.0D0                                                  PIP12590
32.0D0    2300.0D0                                                  PIP12600
-1.0D0 0.0D0                                                        PIP12610
/*                                                                  PIP12620
//                                                                  PIP12630
```

```
//WALLACE JOB (2027,0084),'THESIS N = 6',CLASS=C                    PIP00010
//*MAIN LINES=(20)                                                  PIP00020
//  EXEC FRTXCLGP,IMSL=DP,REGION.GO=2048K                           PIP00030
//FORT.SYSIN DD *                                                   PIP00040
C                                                                   PIP00050
C*******************************************************************PIP00060
C                                                                   PIP00070
C       THIS PROGRAM WAS DEVELOPED TO PERFORM A NUMERICAL ANALYSIS  PIP00080
C                                                                   PIP00090
C       OF THREE DIMENSIONAL PIPE FLOW STABILITY FOR AXIAL WAVE     PIP00100
C                                                                   PIP00110
C       NUMBER N = 6. THE PROGRAM IS DIVIDED INTO EIGHT PARTS.      PIP00120
C                                                                   PIP00130
C       THE FIRST PART COMPUTES THE CENTRAL FINITE DIFFERENCE       PIP00140
C                                                                   PIP00150
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT THE    PIP00160
C                                                                   PIP00170
C       INTERIOR MESH POINTS ALONG THE PIPE RADIUS. IN THE SECOND   PIP00180
C                                                                   PIP00190
C       PART,  PRE-COMPUTED NON-CENTRAL FINITE DIFFERENCE           PIP00200
C                                                                   PIP00210
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT POINTS PIP00220
C                                                                   PIP00230
C       NEAR THE AXIS AND WALL ARE READ IN AS DATA. THE THIRD PART  PIP00240
C                                                                   PIP00250
C       COMPUTES THE VORTICITY TRANSPORT EQUATION COEFFICIENTS OF   PIP00260
C                                                                   PIP00270
C       P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES AT ALL RADIAL PIP00280
C                                                                   PIP00290
C       MESH POINTS (STATIONS). THE FOURTH PART COMPUTES THE A AND B PIP00300
C                                                                   PIP00310
C       MATRIX ELEMENTS,  WHICH MAKE-UP THE EIGENSYSTEM TO BE SOLVED PIP00320
C                                                                   PIP00330
C       IN THE FORMAT,  A * X = GAMMA * B * X.  IN THE FIFTH PART   PIP00340
C                                                                   PIP00350
C       THE A AND B MATRICES COMPRISING THE EIGENSYSTEM ARE SOLVED  PIP00360
C                                                                   PIP00370
C       IN THE SUBROUTINE EIGZC.  THE RESULTING EIGENVALUES AND     PIP00380
C                                                                   PIP00390
C       CORRESPONDING EIGENVECTOR REPRESENT GAMMA AND THE COLUMN    PIP00400
C                                                                   PIP00410
C       VECTOR X,  RESPECTIVELY, AND ARE SOLUTIONS OF THE GOVERNING PIP00420
C                                                                   PIP00430
C       PARTIAL DIFFERENTIAL EQUATIONS.  THE SIXTH PART VERIFIES    PIP00440
C                                                                   PIP00450
C       THE SOLUTION OF THE PDE AND THE MAGNITUDE OF THE RESIDUAL   PIP00460
C                                                                   PIP00470
C       IS DETERMINED.  THE SEVENTH PART COMPUTES THE NORMALIZED    PIP00480
```

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C                                                                       PIP00490
C       AXIAL PERTURBATION VELOCITY U(R).  THE EIGHTH AND LAST          PIP00500
C       PART PLOTS THE PERTURBATION VELOCITY VERSUS PIPE RADIUS.        PIP00510
C       REQUIRED SUBROUTINES ARE INCLUDED AT THE END OF THE PROGRAM     PIP00520
C       LISTING.  THIS IS A GENERALIZED PROGRAM FOR VARIABLE SIZED      PIP00530
C       MESH UP TO A MAXIMUM MESH SIZE OF 50.  BECAUSE OF THE           PIP00540
C       LARGE AMOUNT OF MEMORY REQUIRED TO RUN THIS PROGRAM,  IT        PIP00550
C       CAN ONLY BE RUN IN THE BATCH MODE (MVS).  AVERAGE CPU TIME      PIP00560
C       REQUIRED FOR ONE SET OF DATA,  I.E. ONE AXIAL WAVE NUMBER       PIP00570
C       AND ONE REYNOLDS NUMBER,  IS 61 SECONDS.                        PIP00580
C*************************************************************        PIP00590
C                                                                       PIP00600
C       DEFINE PROGRAM VARIABLES AND ARRAYS.                            PIP00610
C                                                                       PIP00620
      COMPLEX*16 A(100,100),B(100,100)                                  PIP00630
      COMPLEX*16 ZERO,ONE,AAXISQ(4,3,6),AWALLQ(4,3,6),A1DERQ(7)         PIP00640
      COMPLEX*16 A2DERQ(7),A3DERQ(7),A4DERQ(7),A0DER                    PIP00650
      COMPLEX*16 D3Q1(50),D2Q1(50),D1Q1(50),D0Q1(50)                    PIP00660
      COMPLEX*16 D4Q2(50),D3Q2(50),D2Q2(50),D1Q2(50),D0Q2(50)           PIP00670
      COMPLEX*16 E1Q1(50),E0Q1(50),E2Q2(50),E1Q2(50),E0Q2(50)           PIP00680
      COMPLEX*16 AAXISP(2,2,5),AWALLP(2,2,5),A1DERP(5),A2DERP(5)        PIP00690
      COMPLEX*16 D2P1(50),D1P1(50),D0P1(50),D2P2(50),D1P2(50)           PIP00700
      COMPLEX*16 D0P2(50),E0P1(50),E0P2(50)                             PIP00710
      COMPLEX*16 EIGA(100),EIGB(100),EIGVEC(100,100),EIGVAL(100)        PIP00720
      COMPLEX*16 AA(100,100),BB(100,100),SIXIMG,FOUR                    PIP00730
      COMPLEX*16 Q(50),DQ(50),P(50),J(52),RAD(52),UNORM(52)             PIP00740
      COMPLEX*16 ERROR(100),WK(1,1),SUMA,SUMB                           PIP00750
      REAL*8 GAMR(100),GAMMAX,DMESH                                     PIP00760
      REAL*8 H4TH,H3RD,H2ND,H1ST,HTEMP,ATEMP,AATEMP,WATEMP,R,RE,ALPHA   PIP00770
      REAL*8 TEMPR,TEMPI,SIGN,R2,R3,R4,R5                               PIP00780
      REAL*8 AL2,AL12,AL14,AL24,ALSQ,ALSQ2,ALSQ6,ALSQ9,ALSQ24,ALSQ36    PIP00790
      REAL*8 ALSQ57,AL4TH,AL3RD2                                        PIP00800
      REAL*8 UAMP(52),AMPMAX,AMTEMP,AMNOR(52),URDP(52),UIDP(52)         PIP00810
      REAL*4 RADR(52),UR(52),UI(52),AMPNOR(52)                          PIP00820
      REAL*4 X1(2),Y1(2),ALPLOT,REPLOT,GAMPLT,RMESH                     PIP00830
      INTEGER MESH,M,MM,M2,IMAX,MP,MMP,MESH2,NGAM                       PIP00840
C                                                                       PIP00850
C       INITIALIZE THE COEFFICIENT ARRAYS AND SEVERAL CONSTANTS         PIP00860
C                                                                       PIP00870
```

```
      DATA ZERO/(0.0D0,0.0D0)/,ONE/(1.0D0,0.0D0)/,FOUR/(4.0D0,0.0D0)/   PIP00970
      DATA SIXIMG/(0.0D0,6.0D0)/                                        PIP00980
      DATA AAXISQ/72*(0.0D0,0.0D0)/,AWALLQ/72*(0.0D0,0.0D0)/            PIP00990
      DATA A1DERQ/7*(0.0D0,0.0D0)/,A2DERQ/7*(0.0D0,0.0D0)/              PIP01000
      DATA A3DERQ/7*(0.0D0,0.0D0)/,A4DERQ/7*(0.0D0,0.0D0)/              PIP01010
      DATA AAXISP/20*(0.0D0,0.0D0)/,AWALLP/20*(0.0D0,0.0D0)/            PIP01020
      DATA A1DERP/5*(0.0D0,0.0D0)/,A2DERP/5*(0.0D0,0.0D0)/              PIP01030
      DATA X1(1)/0.0/,X1(2)/1.0/,Y1(1)/0.0/,Y1(2)/0.0/                  PIP01040
      DATA MESH/50/                                                     PIP01050
C                                                                       PIP01060
C*****************************************************************    PIP01070
C                                                                       PIP01080
C                              PART I                                   PIP01090
C                                                                       PIP01100
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FOURTH ORDER       PIP01110
C                                                                       PIP01120
C        DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                    PIP01130
C                                                                       PIP01140
      DMESH = DFLOAT(MESH)                                              PIP01150
      H4TH = DMESH ** 4                                                 PIP01160
      ATEMP = (-1.0D0/6.0D0) * H4TH                                     PIP01170
      A4DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01180
      A4DERQ(7) = A4DERQ(1)                                             PIP01190
      ATEMP = 2.0D0 * H4TH                                              PIP01200
      A4DERQ(2) = DCMPLX(ATEMP,0.0D0)                                   PIP01210
      A4DERQ(6) = A4DERQ(2)                                             PIP01220
      ATEMP = -6.5D0 * H4TH                                             PIP01230
      A4DERQ(3) = DCMPLX(ATEMP,0.0D0)                                   PIP01240
      A4DERQ(5) = A4DERQ(3)                                             PIP01250
      ATEMP = (56.0D0/6.0D0) * H4TH                                     PIP01260
      A4DERQ(4) = DCMPLX(ATEMP,0.0D0)                                   PIP01270
C                                                                       PIP01280
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THIRD ORDER        PIP01290
C                                                                       PIP01300
C        DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                    PIP01310
C                                                                       PIP01320
      H3RD = DMESH ** 3                                                 PIP01330
      ATEMP = 0.125D0 * H3RD                                            PIP01340
      A3DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01350
      A3DERQ(7) = -A3DERQ(1)                                            PIP01360
      A3DERQ(6) = DCMPLX(H3RD,0.0D0)                                    PIP01370
      A3DERQ(2) = -A3DERQ(6)                                            PIP01380
      ATEMP = (13.0D0/8.0D0) * H3RD                                     PIP01390
      A3DERQ(3) = DCMPLX(ATEMP,0.0D0)                                   PIP01400
      A3DERQ(5) = -A3DERQ(3)                                            PIP01410
C                                                                       PIP01420
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR SECOND ORDER       PIP01430
C                                                                       PIP01440
```

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C          DERIVATIVES OF P(R) AND Q(R)  AT INTERIOR MESH POINTS.          PIP01450
C                                                                         PIP01460
      H2ND = DMESH ** 2                                                   PIP01470
      ATEMP = (-1.0D0/12.0D0) * H2ND                                      PIP01480
      A2DERQ(2)  = DCMPLX(ATEMP,0.0D0)                                    PIP01490
      A2DERQ(6)  = A2DERQ(2)                                              PIP01500
      ATEMP = (16.0D0/12.0D0) * H2ND                                      PIP01510
      A2DERQ(3)  = DCMPLX(ATEMP,0.0D0)                                    PIP01520
      A2DERQ(5)  = A2DERQ(3)                                              PIP01530
      ATEMP = -2.5D0 * H2ND                                               PIP01540
      A2DERQ(4)  = DCMPLX(ATEMP,0.0D0)                                    PIP01550
      A2DERP(1)  = A2DERQ(2)                                              PIP01560
      A2DERP(2)  = A2DERQ(3)                                              PIP01570
      A2DERP(3)  = A2DERQ(4)                                              PIP01580
      A2DERP(4)  = A2DERQ(5)                                              PIP01590
      A2DERP(5)  = A2DERQ(5)                                              PIP01600
C                                                                         PIP01610
CCC        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FIRST ORDER        PIP01620
CCCC                                                                      PIP01630
CCC        DERIVATIVES OF P(R) AND Q(R)  AT INTERIOR MESH POINTS.         PIP01640
C                                                                         PIP01650
      H1ST = DMESH                                                        PIP01660
      ATEMP = (1.0D0/12.0D0) * H1ST                                       PIP01670
      A1DERQ(2)  = DCMPLX(ATEMP,0.0D0)                                    PIP01680
      A1DERQ(6)  = -A1DERQ(2)                                             PIP01690
      ATEMP = (-8.0D0/12.0D0) * H1ST                                      PIP01700
      A1DERQ(3)  = DCMPLX(ATEMP,0.0D0)                                    PIP01710
      A1DERQ(5)  = -A1DERQ(3)                                             PIP01720
      A1DERP(1)  = A1DERQ(2)                                              PIP01730
      A1DERP(2)  = A1DERQ(3)                                              PIP01740
      A1DERP(4)  = A1DERQ(5)                                              PIP01750
      A1DERP(5)  = A1DERQ(5)                                              PIP01760
C                                                                         PIP01770
CC*************************************************************************PIP01780
CCC                                                                       PIP01790
CCC                          PART II                                      PIP01800
CCC                                                                       PIP01810
CCC        READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR   PIP01820
CCC                                                                       PIP01830
CCC        ALL DERIVATIVES OF P(R) AT POINTS NEAR THE AXIS.               PIP01840
CCC                                                                       PIP01850
CCC        BOUNDARY CONDITION AT THE AXIS IS: P(0) = 0.                   PIP01860
C                                                                         PIP01870
   41 READ(5,49) I,J,K                                                    PIP01880
      IF(I.EQ.9) GO TO 44                                                 PIP01890
      HTEMP = DMESH ** I                                                  PIP01900
      DO 42 KKK = 1,K                                                     PIP01910
      READ(5,*)  AATEMP                                                   PIP01920
```

```
      AATEMP = AATEMP * HTEMP                                          PIP01930
      AAXISP(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                           PIP01940
   42 CONTINUE                                                         PIP01950
      GO TO 41                                                         PIP01960
   44 CONTINUE                                                         PIP01970
C                                                                      PIP01980
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP01990
CCC                                                                    PIP02000
CCC      ALL DERIVATIVES OF P(R) AT POINTS NEAR THE WALL.              PIP02010
CCC                                                                    PIP02020
CCC      BOUNDARY CONDITION AT THE WALL IS:  P(1) = 0                  PIP02030
C                                                                      PIP02040
   45 READ(5,49) I,J,K                                                 PIP02050
      IF(I.EQ.9) GO TO 47                                              PIP02060
      HTEMP = DMESH ** I                                               PIP02070
      JJ = 3 - J                                                       PIP02080
      SIGN = 1.0D0                                                     PIP02090
      IF(I.EQ.1) SIGN = -SIGN                                          PIP02100
      DO 46 KKK = 1,K                                                  PIP02110
      KK = 6 - KKK                                                     PIP02120
      READ(5,*) WATEMP                                                 PIP02130
      WATEMP = WATEMP * HTEMP * SIGN                                   PIP02140
      AWALLP(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                           PIP02150
   46 CONTINUE                                                         PIP02160
      GO TO 45                                                         PIP02170
   47 CONTINUE                                                         PIP02180
   49 FORMAT(I1,1X,I1,1X,I1)                                           PIP02190
C                                                                      PIP02200
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP02210
CCC                                                                    PIP02220
CCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE AXIS.              PIP02230
CCC                                                                    PIP02240
CCC      BOUNDARY CONDITIONS AT THE AXIS ARE: DQ(0) = 0 AND Q(0) = 0.  PIP02250
C                                                                      PIP02260
   51 READ(5,49) I,J,K                                                 PIP02270
      IF(I.EQ.9) GO TO 54                                              PIP02280
      HTEMP = DMESH ** I                                               PIP02290
      DO 52 KKK = 1,K                                                  PIP02300
      READ(5,*) AATEMP                                                 PIP02310
      AATEMP = AATEMP * HTEMP                                          PIP02320
      AAXISQ(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                           PIP02330
   52 CONTINUE                                                         PIP02340
      GO TO 51                                                         PIP02350
   54 CONTINUE                                                         PIP02360
C                                                                      PIP02370
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP02380
CCC                                                                    PIP02390
C        ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE WALL.              PIP02400
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C                                                                       PIP02410
CC        BOUNDARY CONDITIONS AT THE WALL ARE: DQ(1) = 0 AND Q(1) = 0.  PIP02420
C                                                                       PIP02430
   55 READ(5,49) I,J,K                                                  PIP02440
      IF(I.EQ.9) GO TO 57                                               PIP02450
      HTEMP = DMESH ** I                                                PIP02460
      JJ = 4 - J                                                        PIP02470
      SIGN = 1.0D0                                                      PIP02480
      IF(I.EQ.1) SIGN = -SIGN                                           PIP02490
      IF(I.EQ.3) SIGN = -SIGN                                           PIP02500
      DO 56 KKK = 1,K                                                   PIP02510
      KK = 7 - KKK                                                      PIP02520
      READ(5,*) WATEMP                                                  PIP02530
      WATEMP = WATEMP * HTEMP * SIGN                                    PIP02540
      AWALLQ(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                            PIP02550
   56 CONTINUE                                                          PIP02560
      GO TO 55                                                          PIP02570
   57 CONTINUE                                                          PIP02580
C                                                                       PIP02590
CCC       REARRANGE SOME OF THE DATA FOR DQ(R) AND D2Q(R) IN THE        PIP02600
CCC                                                                     PIP02610
CC        AAXISQ AND AWALLQ COEFFICIENT MATRICES.                       PIP02620
C                                                                       PIP02630
      DO 58 I = 1,6                                                     PIP02640
      AAXISQ(2,3,I) = A2DERQ(I + 1)                                     PIP02650
      AAXISQ(1,3,I) = A1DERQ(I + 1)                                     PIP02660
      AWALLQ(2,1,I) = A2DERQ(I)                                         PIP02670
      AWALLQ(1,1,I) = A1DERQ(I)                                         PIP02680
   58 CONTINUE                                                          PIP02690
C                                                                       PIP02700
CC        READ IN ALPHA AND REYNOLDS NR                                 PIP02710
C                                                                       PIP02720
      IPLOT = 0                                                         PIP02730
   19 CONTINUE                                                          PIP02740
      READ(5,*) ALPHA,RE                                                PIP02750
      IF(ALPHA.LT.0.0D0) GO TO 180                                      PIP02760
C                                                                       PIP02770
C*************************************************************************PIP02780
C                                                                       PIP02790
CCC                            PART III                                 PIP02800
CCC                                                                     PIP02810
CC        COMPUTATION OF THE VORTICITY TRANSPORT EQUATION COEFFICIENTS  PIP02820
CCC                                                                     PIP02830
CCC       FOR P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES.           PIP02840
CCC                                                                     PIP02850
CC        PRE-COMPUTE CONSTANTS INVOLVING ALPHA IN THE EQUATIONS.       PIP02860
C                                                                       PIP02870
      AL2 = ALPHA + ALPHA                                               PIP02880
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      AL12 = 12.0D0 * ALPHA                                                  PIP02890
      AL14 = 14.0D0 * ALPHA                                                  PIP02900
      AL24 = 24.0D0 * ALPHA                                                  PIP02910
      ALSQ = ALPHA * ALPHA                                                   PIP02920
      ALSQ2 = ALSQ + ALSQ                                                    PIP02930
      AL4TH = ALSQ * ALSQ                                                    PIP02940
      ALSQ6 = 6.0D0 * ALSQ                                                   PIP02950
      ALSQ9 = 9.0D0 * ALSQ                                                   PIP02960
      ALSQ24 = 24.0D0 * ALSQ                                                 PIP02970
      ALSQ36 = 36.0D0 * ALSQ                                                 PIP02980
      ALSQ57 = 57.0D0 * ALSQ                                                 PIP02990
      AL3RD2 = 2.0D0 * ALSQ * ALPHA                                          PIP03000
C                                                                            PIP03010
CC       COMPUTE THE INCREMENTAL PIPE RADIUS AT INTERIOR MESH POINTS.        PIP03020
C                                                                            PIP03030
      DO 60 I = 1,MESH                                                       PIP03040
      R = (DFLOAT(I) - 0.5D0) / DMESH                                        PIP03050
      R2 = R * R                                                             PIP03060
      R3 = R2 * R                                                            PIP03070
      R4 = R3 * R                                                            PIP03080
      R5 = R4 * R                                                            PIP03090
      RAD(I) = DCMPLX(R,0.0D0)                                               PIP03100
C                                                                            PIP03110
CCC      VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)        PIP03120
CCCC                                                                         PIP03130
CCC      FIRST EQUATION  FIRST QUADRANT                                      PIP03140
C                                                                            PIP03150
      TEMPR = (R2 * (36.0D0 + ALSQ * R2)) / RE                               PIP03160
      D2P1(I) = DCMPLX(TEMPR,0.0D0)                                          PIP03170
      TEMPR = ((252.0D0 * R) + (ALSQ9 * R2)) / RE                            PIP03180
      D1P1(I) = DCMPLX(TEMPR,0.0D0)                                          PIP03190
      TEMPR = -(972.0D0 + (ALSQ57 * R2) + (AL4TH * R4)) / RE                 PIP03200
      TEMPI = - AL2 * R2 * (36.0D0 - (36.0D0 * R2) + (ALSQ * (R2 - R4)))     PIP03210
      D0P1(I) = DCMPLX(TEMPR,TEMPI)                                          PIP03220
      TEMPR = (36.0D0 * R2) + (ALSQ * R4)                                    PIP03230
      E0P1(I) = DCMPLX(TEMPR,0.0D0)                                          PIP03240
C                                                                            PIP03250
CCC      V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R) AND D3Q(R)               PIP03260
CCCC                                                                         PIP03270
CCC      FIRST EQUATION  SECOND QUADRANT                                     PIP03280
C                                                                            PIP03290
      TEMPI = (6.0D0 * R2) / RE                                              PIP03300
      D3Q1(I) = DCMPLX(0.0D0,TEMPI)                                          PIP03310
      TEMPI = (66.0D0 * R) / RE                                              PIP03320
      D2Q1(I) = DCMPLX(0.0D0,TEMPI)                                          PIP03330
      TEMPR = AL12 *(R2 - R4)                                                PIP03340
      TEMPI = - (42.0D0 + (ALSQ6 * R2)) / RE                                 PIP03350
      D1Q1(I) = DCMPLX(TEMPR,TEMPI)                                          PIP03360
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      TEMPR = AL24 * ((2.0D0 * R) - R3)                                 PIP03370
      TEMPI = - ((768.0D0 / R) + (ALSQ36 * R)) / RE                     PIP03380
      D0Q1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03390
      TEMPI = 6.0D0 * R2                                                PIP03400
      E1Q1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03410
      TEMPI = 24.0D0 * R                                                PIP03420
      E0Q1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03430
C                                                                       PIP03440
CCC      VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)   PIP03450
CCCC                                                                    PIP03460
CC       SECOND EQUATION  THIRD QUADRANT                                PIP03470
C                                                                       PIP03480
      TEMPI = - (24.0D0 * R2) / RE                                      PIP03490
      D2P2(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03500
      TEMPI = - (168.0D0 * R) / RE                                      PIP03510
      D1P2(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03520
      TEMPR = - AL24 * (R2 - R4)                                        PIP03530
      TEMPI = (648.0D0 + (ALSQ24 * R2)) / RE                            PIP03540
      D0P2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03550
      TEMPI = - 12.0D0 * R2                                             PIP03560
      E0P2(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03570
C                                                                       PIP03580
CCC      V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R), D3Q(R) AND D4Q(R)  PIP03590
CCCCC                                                                   PIP03600
CC       SECOND EQUATION  FOURTH QUADRANT                               PIP03610
C                                                                       PIP03620
      TEMPR = - R3 / RE                                                 PIP03630
      D4Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03640
      TEMPR = -(14.0D0 * R2) / RE                                       PIP03650
      D3Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03660
      TEMPR = (R * (21.0D0 + (ALSQ2 * R2)))/ RE                         PIP03670
      TEMPI = AL2 * (R3 - R5)                                           PIP03680
      D2Q2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03690
      TEMPR = (315.0D0 + (ALSQ14 * R2)) / RE                            PIP03700
      TEMPI = AL14 * (R2 - R4)                                          PIP03710
      D1Q2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03720
      TEMPR = - ((1152.0D0/R) + (ALSQ56 * R) + (AL4TH * R3)) / RE       PIP03730
      TEMPI = -((56.0D0 * ALPHA * (R - R3))+(AL3RD2 * (R3 - R5)))       PIP03740
      D0Q2(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03750
      TEMPR = - R3                                                      PIP03760
      E2Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03770
      TEMPR = -7.0D0 * R2                                               PIP03780
      E1Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03790
      TEMPR = (28.0D0 * R) + (ALSQ * R3)                                PIP03800
      E0Q2(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03810
   60 CONTINUE                                                          PIP03820
C                                                                       PIP03830
C*************************************************************         PIP03840
```

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C                                                                       PIP03850
CCC                                PART IV                              PIP03860
CCC                                                                     PIP03870
CCC        COMPUTATION OF THE A AND B MATRIX ELEMENTS.                  PIP03880
CCCC                                                                    PIP03890
CCC        INITIALIZE THE A AND B MATRICES                              PIP03900
C                                                                       PIP03910
      DO 198 I = 1,100                                                  PIP03920
      DO 198 J = 1,100                                                  PIP03930
      A(I,J) = ZERO                                                     PIP03940
      B(I,J) = ZERO                                                     PIP03950
  198 CONTINUE                                                          PIP03960
C                                                                       PIP03970
CCC        A AND B MATRIX ELEMENTS IN TERMS OF P(R)                     PIP03980
CCC                                                                     PIP03990
CCC        FIRST EQUATION  FIRST QUADRANT                               PIP04000
C                                                                       PIP04010
      DO 200 J = 1,2                                                    PIP04020
      DO 200 K = 1,5                                                    PIP04030
      AODER = ZERO                                                      PIP04040
      IF (K.EQ.J) AODER = ONE                                           PIP04050
      A(J,K) = AAXISP(2,J,K)*D2P1(J) + AAXISP(1,J,K)*D1P1(J) +          PIP04060
     1         AODER*D0P1(J)                                            PIP04070
      B(J,K) = AODER*E0P1(J)                                            PIP04080
  200 CONTINUE                                                          PIP04090
      MP = MESH - 2                                                     PIP04100
      DO 210 J = 3,MP                                                   PIP04110
      DO 210 K = 1,5                                                    PIP04120
      L = J - 3 + K                                                     PIP04130
      AODER = ZERO                                                      PIP04140
      IF(L.EQ.J) AODER = ONE                                            PIP04150
      A(J,L) = A2DERP(K)*D2P1(J) + A1DERP(K)*D1P1(J) + AODER*D0P1(J)    PIP04160
      B(J,L) = AODER*E0P1(J)                                            PIP04170
  210 CONTINUE                                                          PIP04180
      MMP = MP - 3                                                      PIP04190
      DO 220 J = 1,2                                                    PIP04200
      L = J + MP                                                        PIP04210
      DO 220 K = 1,5                                                    PIP04220
      N = MMP + K                                                       PIP04230
      AODER = ZERO                                                      PIP04240
      IF(N.EQ.L) AODER = ONE                                            PIP04250
      A(L,N) = AWALLP(2,J,K)*D2P1(L) + AWALLP(1,J,K)*D1P1(L) +          PIP04260
     1         AODER*D0P1(L)                                            PIP04270
      B(L,N) = AODER*E0P1(L)                                            PIP04280
  220 CONTINUE                                                          PIP04290
C                                                                       PIP04300
CCC        A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                     PIP04310
C                                                                       PIP04320
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C         FIRST EQUATION  SECOND QUADRANT                                  PIP04330
C                                                                           PIP04340
      DO 250 J = 1,3                                                        PIP04350
      DO 250 K = 1,6                                                        PIP04360
      KK = K + MESH                                                         PIP04370
      AODER = ZERO                                                          PIP04380
      IF(K.EQ.J) AODER = ONE                                                PIP04390
      A(J,KK) =  AAXISQ(3,J,K)*D3Q1(J)  + AAXISQ(2,J,K)*D2Q1(J)  +          PIP04400
     1           AAXISQ(1,J,K)*D1Q1(J)  + AODER*D0Q1(J)                     PIP04410
      B(J,KK) =  AAXISQ(1,J,K)*E1Q1(J)  + AODER*E0Q1(J)                     PIP04420
  250 CONTINUE                                                              PIP04430
      M = MESH - 3                                                          PIP04440
      DO 260 J = 4,M                                                        PIP04450
      DO 260 K = 1,7                                                        PIP04460
      L = J - 4 + K                                                         PIP04470
      LL = L + MESH                                                         PIP04480
      AODER = ZERO                                                          PIP04490
      IF(L.EQ.J) AODER = ONE                                                PIP04500
      A(J,LL) =  A3DERQ(K)*D3Q1(J)  + A2DERQ(K)*D2Q1(J)  +                  PIP04510
     1           A1DERQ(K)*D1Q1(J)  + AODER*D0Q1(J)                         PIP04520
      B(J,LL) =  A1DERQ(K)*E1Q1(J)  + AODER*E0Q1(J)                         PIP04530
  260 CONTINUE                                                              PIP04540
      MM = M - 3                                                            PIP04550
      DO 265 J = 1,3                                                        PIP04560
      L = J + M                                                             PIP04570
      DO 265 K = 1,6                                                        PIP04580
      KK = 7 - K                                                            PIP04590
      N = MM + K                                                            PIP04600
      NN = N + MESH                                                         PIP04610
      AODER = ZERO                                                          PIP04620
      IF(N.EQ.L) AODER = ONE                                                PIP04630
      A(L,NN) =  AWALLQ(3,J,K)*D3Q1(L)  + AWALLQ(2,J,K)*D2Q1(L)  +          PIP04640
     1           AWALLQ(1,J,K)*D1Q1(L)  + AODER*D0Q1(L)                     PIP04650
      B(L,NN) =  AWALLQ(1,J,K)*E1Q1(L)  + AODER*E0Q1(L)                     PIP04660
  265 CONTINUE                                                              PIP04670
C                                                                           PIP04680
CC        A AND B MATRIX ELEMENTS IN TERMS OF P(R)                          PIP04690
CCC                                                                         PIP04700
CCC       SECOND EQUATION  THIRD QUADRANT                                   PIP04710
C                                                                           PIP04720
      DO 270 J = 1,2                                                        PIP04730
      JJ = J + MESH                                                         PIP04740
      DO 270 K = 1,5                                                        PIP04750
      AODER = ZERO                                                          PIP04760
      IF(K.EQ.J) AODER = ONE                                                PIP04770
      A(JJ,K) =  AAXISP(2,J,K)*D2P2(J)  + AAXISP(1,J,K)*D1P2(J)  +          PIP04780
     1           AODER*D0P2(J)                                              PIP04790
      B(JJ,K) =  AODER*E0P2(J)                                              PIP04800
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  270 CONTINUE                                                                  PIP04810
      DO 275 J = 3,MP                                                           PIP04820
      JJ = J + MESH                                                             PIP04830
      DO 275 K = 1,5                                                            PIP04840
      L = J - 3 + K                                                             PIP04850
      A0DER = ZERO                                                              PIP04860
      IF(L.EQ.J) A0DER = ONE                                                    PIP04870
      A(JJ,L) = A2DERP(K)*D2P2(J) + A1DERP(K)*D1P2(J) + A0DER*D0P2(J)           PIP04880
      B(JJ,L) = A0DER*E0P2(J)                                                   PIP04890
  275 CONTINUE                                                                  PIP04900
      DO 280 J = 1,2                                                            PIP04910
      L = J + MP                                                                PIP04920
      LL = L + MESH                                                             PIP04930
      DO 280 K = 1,5                                                            PIP04940
      N = MMP + K                                                               PIP04950
      A0DER = ZERO                                                              PIP04960
      IF (N.EQ.L) A0DER = ONE                                                   PIP04970
      A(LL,N) = AWALLP(2,J,K)*D2P2(L) + AWALLP(1,J,K)*D1P2(L) +                 PIP04980
     1          A0DER*D0P2(L)                                                   PIP04990
      B(LL,N) = A0DER*E0P2(L)                                                   PIP05000
  280 CONTINUE                                                                  PIP05010
C                                                                               PIP05020
CCC       A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                              PIP05030
CCCC                                                                            PIP05040
CCC       SECOND EQUATION  FOURTH QUADRANT                                      PIP05050
C                                                                               PIP05060
      DO 285 J = 1,3                                                            PIP05070
      JJ = J + MESH                                                             PIP05080
      DO 285 K = 1,6                                                            PIP05090
      KK = K + MESH                                                             PIP05100
      A0DER = ZERO                                                              PIP05110
      IF(K.EQ.J) A0DER = ONE                                                    PIP05120
      A(JJ,KK) = AAXISQ(4,J,K)*D4Q2(J) + AAXISQ(3,J,K)*D3Q2(J) +                PIP05130
     1           AAXISQ(2,J,K)*D2Q2(J) + AAXISQ(1,J,K)*D1Q2(J) +                PIP05140
     2           A0DER*D0Q2(J)                                                  PIP05150
      B(JJ,KK) = AAXISQ(2,J,K)*E2Q2(J) + AAXISQ(1,J,K)*E1Q2(J) +                PIP05160
     1           A0DER*E0Q2(J)                                                  PIP05170
  285 CONTINUE                                                                  PIP05180
      DO 290 J = 4,M                                                            PIP05190
      JJ = J + MESH                                                             PIP05200
      DO 290 K = 1,7                                                            PIP05210
      L = J - 4 + K                                                             PIP05220
      LL = L + MESH                                                             PIP05230
      A0DER = ZERO                                                              PIP05240
      IF(L.EQ.J) A0DER = ONE                                                    PIP05250
      A(JJ,LL) = A4DERQ(K)*D4Q2(J) + A3DERQ(K)*D3Q2(J) +                        PIP05260
     1           A2DERQ(K)*D2Q2(J) + A1DERQ(K)*D1Q2(J) + A0DER*D0Q2(J)          PIP05270
      B(JJ,LL) = A2DERQ(K)*E2Q2(J) + A1DERQ(K)*E1Q2(J) + A0DER*E0Q2(J)          PIP05280
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  290 CONTINUE                                                          PIP05290
      DO 295 J = 1,3                                                    PIP05300
      L = J + M                                                         PIP05310
      LL = L + MESH                                                     PIP05320
      DO 295 K = 1,6                                                    PIP05330
      KK = 7 - K                                                        PIP05340
      N = MM + K                                                        PIP05350
      NN = N + MESH                                                     PIP05360
      AODER = ZERO                                                      PIP05370
      IF(N.EQ.L)  AODER = ONE                                           PIP05380
      A(LL,NN) = AWALLQ(4,J,K)*D4Q2(L) + AWALLQ(3,J,K)*D3Q2(L) +        PIP05390
     1           AWALLQ(2,J,K)*D2Q2(L) + AWALLQ(1,J,K)*D1Q2(L) +        PIP05400
     2           AODER*DOQ2(L)                                          PIP05410
      B(LL,NN) = AWALLQ(2,J,K)*E2Q2(L) + AWALLQ(1,J,K)*E1Q2(L) +        PIP05420
     1           AODER*EOQ2(L)                                          PIP05430
  295 CONTINUE                                                          PIP05440
C                                                                       PIP05450
CC*************************************************************       PIP05460
CCCCC                                                                   PIP05470
CCCCC                          PART V                                   PIP05480
CCCCC                                                                   PIP05490
CCCCCC       COMPUTATION OF EIGENVALUES AND RESPECTIVE EIGENVECTORS     PIP05500
CCCCCC       USING THE SUBROUTINE EIGZC WITH THE VORTICITY TRANSPORT    PIP05510
CCCCCC       EQUATIONS IN THE FORM  A * X = GAMMA * B * X.              PIP05520
C                                                                       PIP05530
      DO 140 I = 1,100                                                  PIP05540
      EIGVAL(I) = ZERO                                                  PIP05550
      EIGA(I) = ZERO                                                    PIP05560
      EIGB(I) = ZERO                                                    PIP05570
      DO 140 J = 1,100                                                  PIP05580
      EIGVEC(I,J) = ZERO                                                PIP05590
      AA(I,J) = A(I,J)                                                  PIP05600
      BB(I,J) = B(I,J)                                                  PIP05610
  140 CONTINUE                                                          PIP05620
      WRITE(6,145)                                                      PIP05630
  145 FORMAT(1X,//,1X,'*****************************',                  PIP05640
     1   '*****************************************',                  PIP05650
     2   '*****************************',//)                           PIP05660
      MESH2 = (MESH * 2)                                                PIP05670
      IJOB = 1                                                          PIP05680
      CALL EIGZC(AA,100,BB,100,MESH2,IJOB,EIGA,EIGB,                    PIP05690
     1EIGVEC,100,WK,INEER,IER)                                          PIP05700
      IF(INEER.EQ.0)  WRITE(6,150)                                      PIP05710
  150 FORMAT(1X,////,10X,'ALL EIGENVALUES CONVERGED WITHIN 30 ',        PIP05720
     1   'ITERATIONS',//)                                               PIP05730
      IF(INEER.NE.0)  WRITE(6,155) INEER                                PIP05740
                                                                        PIP05750
                                                                        PIP05760
```

```
  155 FORMAT(1X,////,10X,'FAILED TO CONVERGE ON THE',I4,1X,                PIP05770
     1       'TH EIGENVALUE',//)                                           PIP05780
      DO 165 I = 1,MESH2                                                   PIP05790
      EIGVAL(I) = EIGA(I)/EIGB(I)                                          PIP05800
      GAMR(I) = DREAL(EIGVAL(I))                                           PIP05810
  165 CONTINUE                                                             PIP05820
      NGAM = 1                                                             PIP05830
      GAMMAX = GAMR(1)                                                     PIP05840
      DO 170 I = 2,MESH2                                                   PIP05850
      IF(GAMR(I).LT.GAMMAX) GO TO 169                                      PIP05860
      GAMMAX = GAMR(I)                                                     PIP05870
      NGAM = I                                                             PIP05880
  169 CONTINUE                                                             PIP05890
  170 CONTINUE                                                             PIP05900
      WRITE(6,175) NGAM,GAMMAX,MESH,ALPHA,RE                               PIP05910
  175 FORMAT(1X,/,1X,'LEAST STABLE EIGENVALUE NR',I3,1X,'=',D27.18,4X,     PIP05920
     1 'MESH =',I3,4X,'ALPHA =',D10.3,4X,'REYNOLDS NR =',D17.10,/)         PIP05930
C                                                                          PIP05940
C*************************************************************************PIP05950
C                                                                          PIP05960
C                              PART VI                                     PIP05970
C                                                                          PIP05980
C        VERIFY THAT THE LEAST STABLE EIGENVALUE AND CORRESPONDING         PIP05990
C                                                                          PIP06000
C        EIGENVECTOR SATISFY THE GOVERNING PARTIAL DIFFERENTIAL            PIP06010
C                                                                          PIP06020
C        EQUATIONS.  RESIDUAL ERROR IS OUTPUT AS EPSILON.                  PIP06030
C                                                                          PIP06040
      DO 351 I = 1,MESH2                                                   PIP06050
      SUMA = ZERO                                                          PIP06060
      SUMB = ZERO                                                          PIP06070
      DO 350 J = 1,MESH2                                                   PIP06080
      SUMA = SUMA + ( A(I,J) * EIGVEC(J,NGAM) )                            PIP06090
      SUMB = SUMB + ( B(I,J) * EIGVEC(J,NGAM) )                            PIP06100
  350 CONTINUE                                                             PIP06110
      ERROR(I) = SUMA - ( SUMB * EIGVAL(NGAM) )                            PIP06120
  351 CONTINUE                                                             PIP06130
      WRITE(6,375)                                                         PIP06140
  375 FORMAT(1X,//,15X,'EPSILON = A * X  -  GAMMA * B * X',/)              PIP06150
      WRITE(6,380) (ERROR(I),I = 1,MESH2)                                  PIP06160
  380 FORMAT(1X,2D30.18)                                                   PIP06170
C                                                                          PIP06180
C*************************************************************************PIP06190
C                                                                          PIP06200
C                              PART VII                                    PIP06210
C                                                                          PIP06220
C        COMPUTATION OF AXIAL PERTURBATION VELOCITY  U(R).                 PIP06230
C                                                                          PIP06240
```

```
C         OUTPUT THE EIGENVECTOR CORRESPONDING TO THE LEASE STABLE        PIP06250
CC                                                                        PIP06260
CC        EIGENVALUE,  GAMMA*.                                            PIP06270
C                                                                         PIP06280
      DO 400 I = 1,MESH                                                   PIP06290
      II = I + MESH                                                       PIP06300
      P(I) = EIGVEC(I,NGAM)                                               PIP06310
      Q(I) = EIGVEC(II,NGAM)                                              PIP06320
  400 CONTINUE                                                            PIP06330
      WRITE(6,401) EIGVAL(NGAM)                                           PIP06340
  401 FORMAT(1X,///,5X,'EIGENVECTOR CORRESPONDING TO',                    PIP06350
     1       ' THE LEAST STABLE EIGENVALUE, GAMMA* =',2D27.18,/)          PIP06360
      DO 402 I = 1,MESH                                                   PIP06370
      II = I + MESH                                                       PIP06380
      WRITE(6,404) I,P(I),II,Q(I)                                         PIP06390
  404 FORMAT(2X,I2,2D28.18,2X,I2,2D23.18)                                 PIP06400
  402 CONTINUE                                                            PIP06410
C                                                                         PIP06420
CC        COMPUTATION OF DQ(R)                                            PIP06430
C                                                                         PIP06440
      DO 410 J = 1,3                                                      PIP06450
      DQ(J) = ZERO                                                        PIP06460
      DO 410 K = 1,6                                                      PIP06470
      DQ(J) = DQ(J) +(Q(K) * AAXISQ(1,J,K))                               PIP06480
  410 CONTINUE                                                            PIP06490
C                                                                         PIP06500
      DO 420 J = 4,M                                                      PIP06510
      DQ(J) = ZERO                                                        PIP06520
      DO 420 K = 1,7                                                      PIP06530
      L = J - 4 + K                                                       PIP06540
      DQ(J) = DQ(J) +(Q(L) * A1DERQ(K))                                   PIP06550
  420 CONTINUE                                                            PIP06560
C                                                                         PIP06570
      DO 430 J = 1,3                                                      PIP06580
      L = J + M                                                           PIP06590
      DQ(L) = ZERO                                                        PIP06600
      DO 430 K = 1,6                                                      PIP06610
      KK = 7 - K                                                          PIP06620
      N = MM + K                                                          PIP06630
      DQ(L) = DQ(L) +(Q(N) * AWALLQ(1,J,K))                               PIP06640
  430 CONTINUE                                                            PIP06650
C                                                                         PIP06660
      DO 440 I = 1,MESH                                                   PIP06670
      U(I) = (RAD(I) ** 2) * ((FOUR * Q(I)) + (RAD(I) * DQ(I)) -          PIP06680
     1       (SIXIMG * P(I) * RAD(I)))                                    PIP06690
  440 CONTINUE                                                            PIP06700
C                                                                         PIP06710
CC        REARRANGE DATA FOR PLOT ROUTINE                                 PIP06720
```

```
C                                                                       PIP06730
      M2 = MESH + 2                                                     PIP06740
      DO 450 I = 1,MESH                                                 PIP06750
      II = M2 - I                                                       PIP06760
      III = II - 1                                                      PIP06770
      U(II) = U(III)                                                    PIP06780
      RAD(II) = RAD(III)                                                PIP06790
  450 CONTINUE                                                          PIP06800
      RAD(1) = ZERO                                                     PIP06810
      RAD(M2) = ONE                                                     PIP06820
      U(1) = ZERO                                                       PIP06830
      U(M2) = ZERO                                                      PIP06840
      DO 460 I = 1,M2                                                   PIP06850
      RADR(I) = DREAL(RAD(I))                                           PIP06860
  460 CONTINUE                                                          PIP06870
C                                                                       PIP06880
CCC       COMPUTE THE NORMALIZED PERTURBATION VELOCITY AND              PIP06890
CCC                                                                     PIP06900
CC        NORMALIZED AMPLITUDE AND OUTPUT THE RESULTS.                  PIP06910
C                                                                       PIP06920
      DO 465 I = 1,M2                                                   PIP06930
      UAMP(I) = CDABS(U(I))                                             PIP06940
  465 CONTINUE                                                          PIP06950
      AMPMAX = UAMP(1)                                                  PIP06960
      IMAX = 1                                                          PIP06970
      DO 470 J = 2,M2                                                   PIP06980
      AMTEMP = UAMP(J)                                                  PIP06990
      IF(AMTEMP.LT.AMPMAX) GO TO 471                                    PIP07000
      AMPMAX = UAMP(J)                                                  PIP07010
      IMAX = J                                                          PIP07020
  471 CONTINUE                                                          PIP07030
  470 CONTINUE                                                          PIP07040
      WRITE(6,472)                                                      PIP07050
  472 FORMAT(1X,///,23X,'NORMALIZED PERTURBATION VELOCITY',             PIP07060
     1 24X,'NORMALIZED AMPLITUDE',/)                                    PIP07070
      DO 475 K = 1,M2                                                   PIP07080
      UAMNOR(K) = UAMP(K) / AMPMAX                                      PIP07090
      AMPNOR(K) = UAMNOR(K)                                             PIP07100
      UNORM(K) = U(K) / U(IMAX)                                         PIP07110
      URDP(K) = DREAL(UNORM(K))                                         PIP07120
      UR(K) = URDP(K)                                                   PIP07130
      UIDP(K) = DIMAG(UNORM(K))                                         PIP07140
      UI(K) = UIDP(K)                                                   PIP07150
      KK = K - 1                                                        PIP07160
      WRITE(6,466) KK,URDP(K),UIDP(K),UAMNOR(K)                         PIP07170
  466 FORMAT(1X,I3,1X,2D30.18,5X,D30.18)                                PIP07180
  475 CONTINUE                                                          PIP07190
C                                                                       PIP07200
```

```
C*****************************************************************      PIP07210
C                                                                       PIP07220
C                                PART VIII                              PIP07230
C                                                                       PIP07240
C        PLOT THE NORMALIZED PERTURBATION VELOCITY VS PIPE RADIUS.       PIP07250
C                                                                       PIP07260
      IPLOT = IPLOT + 1                                                 PIP07270
      RMESH = FLOAT(MESH)                                               PIP07280
      ALPLOT = ALPHA                                                    PIP07290
      REPLOT = RE                                                       PIP07300
      GAMPLT = GAMMAX                                                   PIP07310
      CALL PLOTG(RADR,UR,M2,1,1,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0) PIP07320
      CALL PLOTG(RADR,UI,M2,2,1,5,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0) PIP07330
C     CALL PLOTG(RADR,AMPNDR,M2,3,1,2,' ',1,' ',1,0.,1.,-1.,1.,5.,5.)   PIP07340
      CALL PLOTG(X1,Y1,2,3,1,0,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)    PIP07350
      XO = 0.15                                                         PIP07360
      YO = 5.6                                                          PIP07370
      IF(IPLOT.EQ.4) IPLOT = 1                                          PIP07380
      IF(IPLOT.EQ.2) YO = 12.1                                          PIP07390
      IF(IPLOT.EQ.3) YO = 18.6                                          PIP07400
      CALL TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                     PIP07410
      GO TO 19                                                          PIP07420
  180 CONTINUE                                                          PIP07430
      CALL PLOT(0.0,0.0,999)                                            PIP07440
      STOP                                                              PIP07450
      END                                                               PIP07460
C                                                                       PIP07470
C.....................................................................  PIP07480
C                                                                       PIP07490
C        SUBROUTINE TITLE1                                              PIP07500
C                                                                       PIP07510
C        SUBROUTINE TO WRITE TITLE ON PLOTS                             PIP07520
C                                                                       PIP07530
      SUBROUTINE TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)               PIP07540
      RN = 6.0                                                          PIP07550
      YHT = 0.14                                                        PIP07560
      CALL NEWPEN(2)                                                    PIP07570
      CALL SYMBOL(XO,YO,YHT,                                            PIP07580
     1'NORMALIZED PERTURBATION VELOCITY VS RADIUS',0.0,42)              PIP07590
      YHT = 0.08                                                        PIP07600
      DELY = 0.20                                                       PIP07610
      CALL NEWPEN(1)                                                    PIP07620
      YO = YO - 0.8                                                     PIP07630
      XO = XO + 2.5                                                     PIP07640
      XN = XO + .75                                                     PIP07650
      CALL SYMBOL(XO,YO,YHT,'N       = ',0.0,9)                         PIP07660
      CALL NUMBER(XN,YO,YHT,RN,0.0,-1)                                  PIP07670
      YO = YO - DELY                                                    PIP07680
```

```
      CALL SYMBOL(XO,YO,YHT,'MESH    = ',0.0,9)                                  PIP07690
      CALL NUMBER(XN,YO,YHT,RMESH,0.0,-1)                                        PIP07700
      YO = YO - DELY                                                             PIP07710
      CALL SYMBOL(XO,YO,YHT,'ALPHA   = ',0.0,9)                                  PIP07720
      CALL NUMBER(XN,YO,YHT,ALPLOT,0.0,2)                                        PIP07730
      YO = YO - DELY                                                             PIP07740
      CALL SYMBOL(XO,YO,YHT,'REY NR  = ',0.0,9)                                  PIP07750
      CALL NUMBER(XN,YO,YHT,REPLOT,0.0,1)                                        PIP07760
      YO = YO - DELY                                                             PIP07770
      CALL SYMBOL(XO,YO,YHT,'GAMMA*  = ',0.0,9)                                  PIP07780
      CALL NUMBER(XN,YO,YHT,GAMPLT,0.0,4)                                        PIP07790
      YO = YO - 3.4                                                              PIP07800
      CALL SYMBOL(XO,YO,YHT,'OCTAGON = U(REAL)',0.0,17)                          PIP07810
      YO = YO - DELY                                                             PIP07820
      CALL SYMBOL(XO,YO,YHT,'DIAMOND = U(IMAG)',0.0,17)                          PIP07830
      YO = YO - DELY                                                             PIP07840
C     CALL SYMBOL(XO,YO,YHT,'TRIANGLE= U AMPL ',0.0,17)                          PIP07850
CC                                                                               PIP07860
C        WRITE THE AXIS LABLES                                                   PIP07870
C                                                                                PIP07880
      XO = XO - 2.55                                                             PIP07890
      YO = YO + 0.3                                                              PIP07900
      YHT = 0.12                                                                 PIP07910
      CALL SYMBOL(XO,YO,YHT,'NORMALIZED PERTURBATION VELOCITY',90.,32)           PIP07920
      XO = XO + 2.8                                                              PIP07930
      YO = YO - 1.0                                                              PIP07940
      CALL SYMBOL(XO,YO,YHT,'PIPE RADIUS',0.0,11)                                PIP07950
      RETURN                                                                     PIP07960
      END                                                                        PIP07970
/*                                                                               PIP07980
//GO.SYSIN DD *                                                                  PIP07990
2 1 5                                                                            PIP08000
     -0.54166666666665808D+01                                                    PIP08010
      0.24444444444444175D+01                                                    PIP08020
     -0.49999999999994837D+00                                                    PIP08030
      0.95238095238094859D-01                                                    PIP08040
     -0.92592592592588979D-02                                                    PIP08050
2 2 5                                                                            PIP08060
      0.17499999999999245D+01                                                    PIP08070
     -0.27777777777777590D+01                                                    PIP08080
      0.14999999999999178D+01                                                    PIP08090
     -0.14285714285713968 5D+00                                                  PIP08100
      0.92592592592587591 4D-02                                                  PIP08110
1 1 4                                                                            PIP08120
      0.16666666666666586 5D+00                                                  PIP08130
      0.10000000000000022D+01                                                    PIP08140
     -0.29999999999999892D+00                                                    PIP08150
      0.47619047619047638D-01                                                    PIP08160
```

```
1 2 4
     -0.99999999999999973350D+00
      0.16666666666666642980D+00
      0.60000000000000011990D+00
     -0.71428571428571285800D-01
9 0 0
2 1 5
     -0.54166666666666658080D+01
      0.24444444444444441750D+01
     -0.49999999999999948370D+00
      0.95238095238094888590D-01
     -0.92592592592588987900D-02
2 2 5
      0.17499999999999992450D+01
     -0.27777777777777775900D+01
      0.14999999999999991730D+01
     -0.14285714285713968500D+00
      0.92592592592587591400D-02
1 1 4
      0.16666666666666658650D+00
      0.10000000000000000220D+01
     -0.29999999999999998920D+00
      0.47619047619047633800D-01
1 2 4
     -0.99999999999999973350D+00
      0.16666666666666642980D+00
      0.60000000000000011990D+00
     -0.71428571428571285300D-01
9 0 0
4 1 6
      0.11500000000000007140D+03
     -0.38444444444445025500D+02
      0.16400000000000038580D+02
     -0.52244897959184979400D+01
      0.11234567901234950900D+01
     -0.11570247933884755800D+00
4 2 6
     -0.26000000000000080290D+02
      0.17666666666667016000D+02
     -0.11200000000000023020D+02
      0.39591836734695107200D+01
     -0.66666666666670049100D+00
      0.57851239669425180700D-01
4 3 6
      0.10000000000000852650D+01
     -0.62222222222228040800D+01
      0.92000000000004575900D+01
     -0.64489795918368990500D+01
```

```
        0.19876543209876979D+01                                          PIP08650
       -0.16528925619835255D+00                                          PIP08660
3 1 5                                                                    PIP08670
       -0.17499999999999751D+02                                          PIP08680
        0.11111111111107608D+00                                          PIP08690
        0.12000000000001372D+01                                          PIP08700
       -0.34693877551020795D+00                                          PIP08710
        0.43209876543210391D-01                                          PIP08720
3 2 5                                                                    PIP08730
        0.13499999999999680D+02                                          PIP08740
       -0.38888888888888218D+01                                          PIP08750
       -0.14210854715202003 7D-13                                        PIP08760
        0.55102040816326658 4D+00                                        PIP08770
       -0.67901234567901785 9D-01                                        PIP08780
3 3 5                                                                    PIP08790
       -0.15500000000000113 7D+02                                        PIP08800
        0.98888888888891060D+01                                          PIP08810
       -0.60000000000005329D+01                                          PIP08820
        0.14489795 9183674985D+01                                        PIP08830
        0.67901234 5678990936D-01                                        PIP08840
2 1 4                                                                    PIP08850
       -0.46666666666666585D+01                                          PIP08860
        0.21111111111111027D+01                                          PIP08870
       -0.31999999999999993D+00                                          PIP08880
        0.34013605442176852 1D-01                                        PIP08890
2 2 4                                                                    PIP08900
        0.99999999999978584D+00                                          PIP08910
       -0.24444444444444930D+01                                          PIP08920
        0.13200000000000117D+01                                          PIP08930
       -0.81632653061224511 0D-01                                        PIP08940
1 1 3                                                                    PIP08950
        0.24999999999999889D+01                                          PIP08960
        0.22222222222222293D+00                                          PIP08970
       -0.19999999999999978D-01                                          PIP08980
1 2 3                                                                    PIP08990
       -0.45000000000000355D+01                                          PIP09000
        0.13333333333333304D+01                                          PIP09010
        0.18000000000000007D+00                                          PIP09020
9 0 0                                                                    PIP09030
4 1 6                                                                    PIP09040
        0.11500000000000071 4D+03                                        PIP09050
       -0.38444444444450255D+02                                          PIP09060
        0.16400000000003858D+02                                          PIP09070
       -0.52244897959184979 4D+01                                        PIP09080
        0.11234567901234950 9D+01                                        PIP09090
       -0.11570247933884755 3D+00                                        PIP09100
4 2 6                                                                    PIP09110
       -0.26000000000000802 9D+02                                        PIP09120
```

```
  0.17666666666670160D+02                                                PIP09130
 -0.11200000000002302D+02                                                PIP09140
  0.39591836734695107 2D+01                                              PIP09150
 -0.66666666666700491D+00                                                PIP09160
  0.57851239669 4251807D-01                                              PIP09170
4 3 6                                                                    PIP09180
  0.10000000000085265D+01                                                PIP09190
 -0.62222222222804 08D+01                                                PIP09200
  0.92000000000045759D+01                                                PIP09210
 -0.64489795918689906D+01                                                PIP09220
  0.19876543209 87697 79D+01                                             PIP09230
 -0.16528925619 8352552D+00                                              PIP09240
3 1 5                                                                    PIP09250
 -0.17499999999999751D+02                                                PIP09260
  0.11111111111107608D+00                                                PIP09270
  0.12000000000001372D+01                                                PIP09280
 -0.34693877551 0207395D+00                                              PIP09290
  0.43209876543 2103911D-01                                              PIP09300
3 2 5                                                                    PIP09310
  0.13499999999999680D+02                                                PIP09320
 -0.38888888888888218D+01                                                PIP09330
 -0.14210854715 20200 37D-13                                             PIP09340
  0.55102040816 3266584D+00                                              PIP09350
 -0.67901234567 9017859D-01                                              PIP09360
3 3 5                                                                    PIP09370
 -0.15500000000000 1137D+02                                              PIP09380
  0.98888888888 8891060D+01                                              PIP09390
 -0.60000000000 0053 29D+01                                              PIP09400
  0.14489795 91836 74985D+01                                             PIP09410
  0.67901234 5678990 935D-01                                             PIP09420
2 1 4                                                                    PIP09430
 -0.46666666666666585D+01                                                PIP09440
  0.21111111111111027D+01                                                PIP09450
 -0.31999999999999993D+00                                                PIP09460
  0.34013605 442 17685 21D-01                                            PIP09470
2 2 4                                                                    PIP09480
  0.99999999 9999978 684D+00                                             PIP09490
 -0.24444444 4444444930D+01                                              PIP09500
  0.13200000 00 000000 117D+01                                           PIP09510
 -0.81632653 0612245 110D-01                                             PIP09520
1 1 3                                                                    PIP09530
  0.24999999 9999999889D+01                                              PIP09540
  0.22222222 2222222293D+00                                              PIP09550
 -0.19999999 9999999973D-01                                              PIP09560
1 2 3                                                                    PIP09570
 -0.45000000 0000000 355D+01                                             PIP09580
  0.13333333 3333333304D+01                                              PIP09590
  0.18000000 0000000007D+00                                              PIP09600
```

```
9 0  0                                                                  PIP09610
0.0D0      1150.0D0                                                     PIP09620
0.5D0      1150.0D0                                                     PIP09630
1.0D0      1150.0D0                                                     PIP09640
2.0D0      1150.0D0                                                     PIP09650
4.0D0      1150.0D0                                                     PIP09660
8.0D0      1150.0D0                                                     PIP09670
16.0D0     1150.0D0                                                     PIP09680
32.0D0     1150.0D0                                                     PIP09690
0.0D0      2300.0D0                                                     PIP09700
0.5D0      2300.0D0                                                     PIP09710
1.0D0      2300.0D0                                                     PIP09720
2.0D0      2300.0D0                                                     PIP09730
4.0D0      2300.0D0                                                     PIP09740
8.0D0      2300.0D0                                                     PIP09750
16.0D0     2300.0D0                                                     PIP09760
32.0D0     2300.0D0                                                     PIP09770
-1.0D0     0.0D0                                                        PIP09780
/*                                                                      PIP09790
//                                                                      PIP09800
```

D. PROGRAM TO COMPUTE THE NONCENTRAL FINITE DIFFERENCE COEFFICIENTS

```
C**********************************************************************
C
C     THIS IS A PROGRAM USED TO COMPUTE THE NONCENTRAL FINITE
C     DIFFERENCE COEFFICIENTS FOR THE DERIVATIVES OF THE FUNCTIONS
C     P(R) AND Q(R) FOR THE THREE CASES INVESTIGATED.  THE DATA
C     IS THEN READ INTO THE MAIN INVESTIGATIVE PROGRAMS FOR N = 0,
C     1 AND 6.  THE DATA IS ANNOTATED AS TO IT'S APPLICATION
C     DEPENDING ON THE BOUNDARY CONDITIONS AT THE AXIS OR WALL.
C     THESE ANNOTATIONS SHOULD BE REMOVED BEFORE THE DATA IS USED.
C     THE PROGRAM IS RUN IN THE VM MODE AND THE DATA IS OUTPUT
C     TO A FILE ON THE USER'S DISK WITH A FILENAME = COEFF AND
C     FILETYPE = DATAOUT2.  THIS IS ACCOMPLISHED WITH THE USE OF
C     SPECIAL EXEC FILE CALLED,  RUNDATA.  THIS EXEC FILE IS
C     LISTED AT THE END OF THE COEFFICIENT PROGRAM.  THE PROPER
C     COMMAND TO RUN THE PROGRAM IS,  RUNDATA COEFF T.
C**********************************************************************
      REAL*8 A6(6,6),FACT,H(6),XJ,XI,A6INV(6,6),WKAREA(60)
      REAL*8 A5(5,5),A4(4,4),A5INV(5,5),A4INV(4,4),A3(3,3),A3INV(3,3)
      REAL*8 C(3,4,6)
      REAL*8 C4X61(4,6),C4X62(4,6),C4X63(4,6)
      REAL*8 C4X51(4,5),C4X52(4,5),C4X53(4,5)
      REAL*8 C3X51(3,5),C3X52(3,5),C3X53(3,5)
      REAL*8 C4X41(4,4),C4X42(4,4)
      REAL*8 C2X41(2,4),C2X42(2,4)
      REAL*8 C3X31(3,3),C3X32(3,3)
      REAL*8 C1X31(1,3),C1X32(1,3)
      REAL*8 C46A61(4,6),C46A62(4,6),C46A63(4,6)
      REAL*8 C45A51(4,5),C45A52(4,5),C45A53(4,5)
      REAL*8 C35A51(3,5),C35A52(3,5),C35A53(3,5)
      REAL*8 C44A41(4,4),C44A42(4,4)
      REAL*8 C24A41(2,4),C24A42(2,4)
      REAL*8 C33A31(3,3),C33A32(3,3)
      REAL*8 C13A31(1,3),C13A32(1,3)
      REAL*8 C2X51(2,5),C2X52(2,5)
```

```
      REAL*8 C1X41(1,4),C1X42(1,4)                                      CDE00490
      REAL*8 C25A51(2,5),C25A52(2,5)                                    CDE00500
      REAL*8 C14A41(1,4),C14A42(1,4)                                    CDE00510
C                                                                       CDE00520
CCCC     THE AA MATRICES ARE 6 X 6, 5 X 5, 4 X 4 & 3 X 3  AND ARE       CDE00530
CCCC                                                                    CDE00540
CCCC     CALLED A6,A5,A4 & A3, RESPECTIVELY.                            CDE00550
C                                                                       CDE00560
      DO 10 I=1,6                                                       CDE00570
      FACT = 1.0D0                                                      CDE00580
      XI = DFLOAT(I)                                                    CDE00590
      H(I) = (2.0D0*XI-1.0D0)/2.0D0                                     CDE00600
      DO 10 J=1,6                                                       CDE00610
      XJ = DFLOAT(J)                                                    CDE00620
      FACT = FACT*(XJ+1.0D0)                                            CDE00630
      A6(I,J) = (H(I)**(J+1))/FACT                                      CDE00640
   10 CONTINUE                                                          CDE00650
   20 FORMAT(1X,D29.18)                                                 CDE00660
C                                                                       CDE00670
CC*****************************************************************   CDE00680
CCCC                                                                    CDE00690
CCCC     A SUBPROGRAM USED TO COMPUTE THE NONCENTRAL DIFFERENCE         CDE00700
CCCC                                                                    CDE00710
CCCC     COEFFICIENTS THE AXIS FOR N = 0.                               CDE00720
CCCC                                                                    CDE00730
CCCC     THE APPLICABLE BOUNDARY CONDITIONS ARE:                        CDE00740
CCCC                                                                    CDE00750
CCCC     DQ(0) = 0 AND D3Q(0) = 0  FOR N = 0 AT THE AXIS.               CDE00760
CCCC                                                                    CDE00770
CC*****************************************************************   CDE00780
CCCC                                                                    CDE00790
CCCC     COMPUTE THE AA MATRICES AND THEIR INVERSES.                    CDE00800
C                                                                       CDE00810
      DO 11 I = 1,6                                                     CDE00820
      A6(I,2) = A6(I,1)                                                 CDE00830
      A6(I,1) = 1.0D0                                                   CDE00840
   11 CONTINUE                                                          CDE00850
      CALL LINV2F(A6,6,6,A6INV,10,WKAREA,IER)                           CDE00860
      DO 150 I = 1,5                                                    CDE00870
      DO 150 J = 1,5                                                    CDE00880
      A5(I,J) = A6(I,J)                                                 CDE00890
  150 CONTINUE                                                          CDE00900
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                           CDE00910
      DO 170 I = 1,4                                                    CDE00920
      DO 170 J = 1,4                                                    CDE00930
      A4(I,J) = A6(I,J)                                                 CDE00940
  170 CONTINUE                                                          CDE00950
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                           CDE00960
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```
      DO 171 I = 1,3                                                    CDE00970
      DO 171 J = 1,3                                                    CDE00980
      A3(I,J) = A6(I,J)                                                 CDE00990
  171 CONTINUE                                                          CDE01000
      CALL LINV2F(A3,3,3,A3INV,10,WKAREA,IER)                           CDE01010
C                                                                       CDE01020
CC       COMPUTE THE CC MATRICES FOR APPROPRIATE POINTS.                CDE01030
C                                                                       CDE01040
      DO 310 K = 1,3                                                    CDE01050
      XK = DFLOAT(K)                                                    CDE01060
      H(K) = (2.0D0*XK - 1.0D0)/2.0D0                                   CDE01070
      C(K,2,1) = 1.0D0                                                  CDE01080
      C(K,3,1) = 0.0D0                                                  CDE01090
      C(K,4,1) = 0.0D0                                                  CDE01100
      C(K,3,2) = 1.0D0                                                  CDE01110
      C(K,4,2) = 0.0D0                                                  CDE01120
      C(K,4,3) = 1.0D0                                                  CDE01130
      KK = 0                                                            CDE01140
      K6 = 6                                                            CDE01150
      DO 330 I = 1,4                                                    CDE01160
      FACT = 1.0D0                                                      CDE01170
      DO 325 J = 1,K6                                                   CDE01180
      JK = J+KK                                                         CDE01190
      XJ = DFLOAT(J)                                                    CDE01200
      FACT = FACT*XJ                                                    CDE01210
      C(K,I,JK) = (H(K)**J)/FACT                                        CDE01220
  325 CONTINUE                                                          CDE01230
      KK = KK+1                                                         CDE01240
      K6 = K6-1                                                         CDE01250
  330 CONTINUE                                                          CDE01260
  310 CONTINUE                                                          CDE01270
      DO 331 K = 1,3                                                    CDE01280
      DO 331 I = 1,4                                                    CDE01290
      C(K,I,2) = C(K,I,1)                                               CDE01300
      C(K,I,1) = 0.0D0                                                  CDE01310
  331 CONTINUE                                                          CDE01320
C                                                                       CDE01330
CC       CONVERT THE 3 DIMENSIONAL CC MATRICES TO APPROPRIATE 2         CDE01340
CC       DIMENSIONAL MATRICES.                                          CDE01350
C                                                                       CDE01360
C                                                                       CDE01370
      DO 200 I = 1,4                                                    CDE01380
      DO 200 J = 1,6                                                    CDE01390
      C4X61(I,J) = C(1,I,J)                                             CDE01400
  200 CONTINUE                                                          CDE01410
      DO 201 I = 1,4                                                    CDE01420
      DO 201 J = 1,6                                                    CDE01430
      C4X62(I,J) = C(2,I,J)                                             CDE01440
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  201 CONTINUE                                                          COE01450
      DO 202 I = 1,4                                                    COE01460
      DO 202 J = 1,6                                                    COE01470
      C4X63(I,J) = C(3,I,J)                                             COE01480
  202 CONTINUE                                                          COE01490
      DO 203 I = 1,4                                                    COE01500
      DO 203 J = 1,5                                                    COE01510
      C4X51(I,J) = C4X61(I,J)                                           COE01520
  203 CONTINUE                                                          COE01530
      DO 204 I = 1,4                                                    COE01540
      DO 204 J = 1,4                                                    COE01550
      C4X41(I,J) = C4X61(I,J)                                           COE01560
  204 CONTINUE                                                          COE01570
      DO 220 I = 1,3                                                    COE01580
      DO 220 J = 1,3                                                    COE01590
      C3X31(I,J) = C4X61(I,J)                                           COE01600
  220 CONTINUE                                                          COE01610
      DO 205 I = 1,4                                                    COE01620
      DO 205 J = 1,5                                                    COE01630
      C4X52(I,J) = C4X62(I,J)                                           COE01640
  205 CONTINUE                                                          COE01650
      DO 206 I = 1,4                                                    COE01660
      DO 206 J = 1,4                                                    COE01670
      C4X42(I,J) = C4X62(I,J)                                           COE01680
  206 CONTINUE                                                          COE01690
      DO 208 I = 1,3                                                    COE01700
      DO 208 J = 1,3                                                    COE01710
      C3X32(I,J) = C4X62(I,J)                                           COE01720
  208 CONTINUE                                                          COE01730
      DO 207 I = 1,4                                                    COE01740
      DO 207 J = 1,5                                                    COE01750
      C4X53(I,J) = C4X63(I,J)                                           COE01760
  207 CONTINUE                                                          COE01770
C                                                                       COE01780
CCCCC     PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER       COE01790
C                                                                       COE01800
CCCCC     CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.         COE01810
C                                                                       COE01820
      CALL VMULFF(C4X61,A5INV,4,6,6,4,6,C46A61,4,IER)                   COE01830
      CALL VMULFF(C4X62,A5INV,4,6,6,4,6,C46A62,4,IER)                   COE01840
      CALL VMULFF(C4X63,A5INV,4,6,6,4,6,C46A63,4,IER)                   COE01850
      CALL VMULFF(C4X51,A5INV,4,5,5,4,5,C45A51,4,IER)                   COE01860
      CALL VMULFF(C4X52,A5INV,4,5,5,4,5,C45A52,4,IER)                   COE01870
      CALL VMULFF(C4X53,A5INV,4,5,5,4,5,C45A53,4,IER)                   COE01880
      CALL VMULFF(C4X41,A4INV,4,4,4,4,4,C44A41,4,IER)                   COE01890
      CALL VMULFF(C4X42,A4INV,4,4,4,4,4,C44A42,4,IER)                   COE01900
      CALL VMULFF(C3X31,A3INV,3,3,3,3,3,C33A31,3,IER)                   COE01910
      CALL VMULFF(C3X32,A3INV,3,3,3,3,3,C33A32,3,IER)                   COE01920
```

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      WRITE(2,210)                                                      COE01930
      WRITE(2,209)                                                      COE01940
  209 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQUCOE01950
     1IRED NEAR THE AXIS FOR N = 0.',//,1X,                             COE01960
     2'THE COEFFICIENTS APPEAR IN THE ',                                COE01970
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X, COE01980
     4'THE APPLICABLE BOUNDARY CONDITIONS ARE:',//,1X,                  COE01990
     5'DQ(0) = 0  AND  D3Q(0) = 0  FOR N = 0  AT THE AXIS.')            COE02000
      WRITE(2,210)                                                      COE02010
  210 FORMAT(1X,//)                                                     COE02020
      ND = 4                                                            COE02030
      NP = 1                                                            COE02040
      NPTS = 6                                                          COE02050
      WRITE(2,221) ND,NP,NPTS                                           COE02060
  221 FORMAT(I1,1X,I1,1X,I1)                                            COE02070
      WRITE(2,20)  (C46A61(4,J),J=1,NPTS)                               COE02080
      NP = NP + 1                                                       COE02090
      WRITE(2,221) ND,NP,NPTS                                           COE02100
      WRITE(2,20)  (C46A62(4,J),J=1,NPTS)                               COE02110
      NP = NP + 1                                                       COE02120
      WRITE(2,221) ND,NP,NPTS                                           COE02130
      WRITE(2,20)  (C46A63(4,J),J=1,NPTS)                               COE02140
      ND = 3                                                            COE02150
      NP = 1                                                            COE02160
      NPTS = NPTS - 1                                                   COE02170
      WRITE(2,221) ND,NP,NPTS                                           COE02180
      WRITE(2,20)  (C45A51(3,J),J=1,NPTS)                               COE02190
      NP = NP + 1                                                       COE02200
      WRITE(2,221) ND,NP,NPTS                                           COE02210
      WRITE(2,20)  (C45A52(3,J),J=1,NPTS)                               COE02220
      NP = NP + 1                                                       COE02230
      WRITE(2,221) ND,NP,NPTS                                           COE02240
      WRITE(2,20)  (C45A53(3,J),J=1,NPTS)                               COE02250
      ND = 2                                                            COE02260
      NP = 1                                                            COE02270
      NPTS = NPTS - 1                                                   COE02280
      WRITE(2,221) ND,NP,NPTS                                           COE02290
      WRITE(2,20)  (C44A41(2,J),J=1,NPTS)                               COE02300
      NP = NP + 1                                                       COE02310
      WRITE(2,221) ND,NP,NPTS                                           COE02320
      WRITE(2,20)  (C44A42(2,J),J=1,NPTS)                               COE02330
      ND = 1                                                            COE02340
      NP = 1                                                            COE02350
      NPTS = NPTS - 1                                                   COE02360
      WRITE(2,221) ND,NP,NPTS                                           COE02370
      WRITE(2,20)  (C33A31(1,J),J=1,NPTS)                               COE02380
      NP = NP + 1                                                       COE02390
      WRITE(2,221) ND,NP,NPTS                                           COE02400
```

```
      WRITE(2,20)  (C33A32(1,J),J=1,NPTS)                                CODE02410
      ND = 9                                                             CODE02420
      NP = 0                                                             CODE02430
      NPTS = 0                                                           CODE02440
      WRITE(2,221) ND,NP,NPTS                                            CODE02450
      WRITE(2,225)                                                       CODE02460
  225 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE Q(0) FOR N = 0.  TAKEN',    CODE02470
     1' FROM THE TOP ROW OF THE A6INV MATRIX.')                          CODE02480
      WRITE(2,20) (A6INV(1,J),J=1,6)                                     CODE02490
C                                                                        CODE02500
C*************************************************************         CODE02510
C                                                                        CODE02520
C        A MODIFIED SUBPROGRAM TO COMPUTE AA MATRIX ELEMENTS FOR         CODE02530
C                                                                        CODE02540
C        THE BOUNDARY CONDITIONS NEAR THE AXIS AND THE WALL FOR          CODE02550
C                                                                        CODE02560
C        N = 0, 1 AND 6.  THE APPLICABLE BOUNDARY CONDITIONS ARE:        CODE02570
C                                                                        CODE02580
C        Q(1) = 0 AND DQ(1) = 0  FOR N = 0 AT THE WALL.                  CODE02590
C                                                                        CODE02600
C        Q(0) = VARIABLE AND DQ(0) = 0  FOR N = 1 AT THE AXIS.           CODE02610
C                                                                        CODE02620
C        Q(1) = -H(0) AND DQ(0) = 2H(0)  FOR N = 1 AT THE WALL.          CODE02630
C                                                                        CODE02640
C        Q(0) = 0 AND DQ(0) = 0  FOR N = 6 AT THE AXIS.                  CODE02650
C                                                                        CODE02660
C        Q(1) = 0 AND DQ(1) = 0  FOR N = 6 AT THE WALL.                  CODE02670
C                                                                        CODE02680
C*************************************************************         CODE02690
C                                                                        CODE02700
C        COMPUTE THE AA MATRICES AND THEIR INVERSES.                     CODE02710
C                                                                        CODE02720
      DO 400 I=1,6                                                       CODE02730
      FACT = 1.0D0                                                       CODE02740
      XI = DFLOAT(I)                                                     CODE02750
      H(I) = (2.0D0*XI-1.0D0)/2.0D0                                      CODE02760
      DO 400 J=1,6                                                       CODE02770
      XJ = DFLOAT(J)                                                     CODE02780
      FACT = FACT*(XJ+1.0D0)                                             CODE02790
      A6(I,J) = (H(I)**(J+1))/FACT                                       CODE02800
  400 CONTINUE                                                           CODE02810
      CALL LINV2F(A6,6,6,A6INV,10,WKAREA,IER)                            CODE02820
      DO 401 I = 1,5                                                     CODE02830
      DO 401 J = 1,5                                                     CODE02840
      A5(I,J) = A6(I,J)                                                  CODE02850
  401 CONTINUE                                                           CODE02860
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                            CODE02870
      DO 402 I = 1,4                                                     CODE02880
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      DO 402 J = 1,4                                                    CODE02890
      A4(I,J) = A6(I,J)                                                 CODE02900
  402 CONTINUE                                                          CODE02910
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                           CODE02920
      DO 403 I = 1,3                                                    CODE02930
      DO 403 J = 1,3                                                    CODE02940
      A3(I,J) = A6(I,J)                                                 CODE02950
  403 CONTINUE                                                          CODE02960
      CALL LINV2F(A3,3,3,A3INV,10,WKAREA,IER)                           CODE02970
C                                                                       CODE02980
CCC      COMPUTE THE CC COEFFICIENT MATRICES FOR APPROPRIATE POINTS.    CODE02990
C                                                                       CODE03000
      DO 405 K = 1,3                                                    CODE03010
      XK = DFLOAT(K)                                                    CODE03020
      H(K) = (2.0D0*XK - 1.0D0)/2.0D0                                   CODE03030
      C(K,2,1) = 1.0D0                                                  CODE03040
      C(K,3,1) = 0.0D0                                                  CODE03050
      C(K,4,1) = 0.0D0                                                  CODE03060
      C(K,3,2) = 1.0D0                                                  CODE03070
      C(K,4,2) = 0.0D0                                                  CODE03080
      C(K,4,3) = 1.0D0                                                  CODE03090
      KK = 0                                                            CODE03100
      K6 = 6                                                            CODE03110
      DO 406 I = 1,4                                                    CODE03120
      FACT = 1.0D0                                                      CODE03130
      DO 407 J = 1,K6                                                   CODE03140
      JK = J+KK                                                         CODE03150
      XJ = DFLOAT(J)                                                    CODE03160
      FACT = FACT*XJ                                                    CODE03170
      C(K,I,JK) = (H(K)**J)/FACT                                        CODE03180
  407 CONTINUE                                                          CODE03190
      KK = KK+1                                                         CODE03200
      K6 = K6-1                                                         CODE03210
  406 CONTINUE                                                          CODE03220
  405 CONTINUE                                                          CODE03230
C                                                                       CODE03240
CCC      CONVERT THE 3 DIMENSIONAL CC MATRICES TO APPROPRIATE 2         CODE03250
CCC                                                                     CODE03260
CCC      DIMENSIONAL MATRICES.                                          CODE03270
C                                                                       CODE03280
      DO 408 I = 1,4                                                    CODE03290
      DO 408 J = 1,6                                                    CODE03300
      C4X61(I,J) = C(1,I,J)                                             CODE03310
  408 CONTINUE                                                          CODE03320
      DO 409 I = 1,4                                                    CODE03330
      DO 409 J = 1,6                                                    CODE03340
      C4X62(I,J) = C(2,I,J)                                             CODE03350
  409 CONTINUE                                                          CODE03360
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      DO 410 I = 1,4                                                    CDE03370
      DO 410 J = 1,6                                                    CDE03380
      C4X63(I,J) = C(3,I,J)                                             CDE03390
  410 CONTINUE                                                          CDE03400
      DO 411 I = 1,3                                                    CDE03410
      DO 411 J = 1,5                                                    CDE03420
      C3X51(I,J) = C4X61(I,J)                                           CDE03430
  411 CONTINUE                                                          CDE03440
      DO 412 I = 1,2                                                    CDE03450
      DO 412 J = 1,4                                                    CDE03460
      C2X41(I,J) = C4X61(I,J)                                           CDE03470
  412 CONTINUE                                                          CDE03480
      DO 413 J = 1,3                                                    CDE03490
      C1X31(1,J) = C4X61(1,J)                                           CDE03500
  413 CONTINUE                                                          CDE03510
      DO 414 I = 1,3                                                    CDE03520
      DO 414 J = 1,5                                                    CDE03530
      C3X52(I,J) = C4X62(I,J)                                           CDE03540
  414 CONTINUE                                                          CDE03550
      DO 415 I = 1,2                                                    CDE03560
      DO 415 J = 1,4                                                    CDE03570
      C2X42(I,J) = C4X62(I,J)                                           CDE03580
  415 CONTINUE                                                          CDE03590
      DO 416 J = 1,3                                                    CDE03600
      C1X32(1,J) = C4X62(1,J)                                           CDE03610
  416 CONTINUE                                                          CDE03620
      DO 417 I = 1,3                                                    CDE03630
      DO 417 J = 1,5                                                    CDE03640
      C3X53(I,J) = C4X63(I,J)                                           CDE03650
  417 CONTINUE                                                          CDE03660
C                                                                       CDE03670
CC        PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER       CDE03680
CCC                                                                     CDE03690
CC        CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.         CDE03700
C                                                                       CDE03710
      CALL VMULFF(C4X61,A5INV,4,6,6,4,6,C46A61,4,IER)                   CDE03720
      CALL VMULFF(C4X62,A6INV,4,6,6,4,6,C46A62,4,IER)                   CDE03730
      CALL VMULFF(C4X63,A5INV,4,6,6,4,6,C46A63,4,IER)                   CDE03740
      CALL VMULFF(C3X51,A5INV,3,5,5,3,5,C35A51,3,IER)                   CDE03750
      CALL VMULFF(C3X52,A5INV,3,5,5,3,5,C35A52,3,IER)                   CDE03760
      CALL VMULFF(C3X53,A5INV,3,5,5,3,5,C35A53,3,IER)                   CDE03770
      CALL VMULFF(C2X41,A4INV,2,4,4,2,4,C24A41,2,IER)                   CDE03780
      CALL VMULFF(C2X42,A4INV,2,4,4,2,4,C24A42,2,IER)                   CDE03790
      CALL VMULFF(C1X31,A3INV,1,3,3,1,3,C13A31,1,IER)                   CDE03800
      CALL VMULFF(C1X32,A3INV,1,3,3,1,3,C13A32,1,IER)                   CDE03810
      WRITE(2,419)                                                      CDE03820
  419 FORMAT(1X,//)                                                     CDE03830
      WRITE(2,418)                                                      CDE03840
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 418 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQUCOE03850
    1IRED NEAR THE BOUNDARIES FOR N = 0, 1 AND 6.',//,1X,                CDE03860
    2'THE COEFFICIENTS APPEAR IN THE ',                                  CDE03870
    3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X,   CDE03880
    4'THE APPLICABLE BOUNDARY CONDITIONS ARE:',//,1X,                    CDE03890
    5'Q(1)  = 0   AND   DQ(1)  = 0   FOR N = 0 AT THE WALL.',//,1X,      CDE03900
    6'Q(0)  = VARIABLE  AND   DQ(0) = 0   FOR N = 1 AT THE AXIS.',//,1X, CDE03910
    7'Q(1)  = -H(0)   AND   DQ(0)  = 2H(0)   FOR N = 1 AT THE WALL.',//,1X,CDE03920
    8'Q(0)  = 0   AND   DQ(0)  = 0   FOR N = 6 AT THE AXIS.',//,1X,      CDE03930
    9'Q(1)  = 0   AND   DQ(1)  = 0   FOR N = 6 AT THE WALL.',//)         CDE03940
      ND = 4                                                             CDE03950
      NP = 1                                                             CDE03960
      NPTS = 6                                                           CDE03970
      WRITE(2,221)  ND,NP,NPTS                                           CDE03980
      WRITE(2,20)   (C46A61(4,J),J=1,NPTS)                               CDE03990
      NP = NP + 1                                                        CDE04000
      WRITE(2,221)  ND,NP,NPTS                                           CDE04010
      WRITE(2,20)   (C46A62(4,J),J=1,NPTS)                               CDE04020
      NP = NP + 1                                                        CDE04030
      WRITE(2,221)  ND,NP,NPTS                                           CDE04040
      WRITE(2,20)   (C46A63(4,J),J=1,NPTS)                               CDE04050
      ND = 3                                                             CDE04060
      NP = 1                                                             CDE04070
      NPTS = NPTS - 1                                                    CDE04080
      WRITE(2,221)  ND,NP,NPTS                                           CDE04090
      WRITE(2,20)   (C35A51(3,J),J=1,NPTS)                               CDE04100
      NP = NP + 1                                                        CDE04110
      WRITE(2,221)  ND,NP,NPTS                                           CDE04120
      WRITE(2,20)   (C35A52(3,J),J=1,NPTS)                               CDE04130
      NP = NP + 1                                                        CDE04140
      WRITE(2,221)  ND,NP,NPTS                                           CDE04150
      WRITE(2,20)   (C35A53(3,J),J=1,NPTS)                               CDE04160
      ND = 2                                                             CDE04170
      NP = 1                                                             CDE04180
      NPTS = NPTS - 1                                                    CDE04190
      WRITE(2,221)  ND,NP,NPTS                                           CDE04200
      WRITE(2,20)   (C24A41(2,J),J=1,NPTS)                               CDE04210
      NP = NP + 1                                                        CDE04220
      WRITE(2,221)  ND,NP,NPTS                                           CDE04230
      WRITE(2,20)   (C24A42(2,J),J=1,NPTS)                               CDE04240
      ND = 1                                                             CDE04250
      NP = 1                                                             CDE04260
      NPTS = NPTS - 1                                                    CDE04270
      WRITE(2,221)  ND,NP,NPTS                                           CDE04280
      WRITE(2,20)   (C13A31(1,J),J=1,NPTS)                               CDE04290
      NP = NP + 1                                                        CDE04300
      WRITE(2,221)  ND,NP,NPTS                                           CDE04310
      WRITE(2,20)   (C13A32(1,J),J=1,NPTS)                               CDE04320
```

```
      ND = 9                                                            CDE04330
      NP = 0                                                            CDE04340
      NPTS = 0                                                          CDE04350
      WRITE(2,221) ND,NP,NPTS                                           CDE04360
      WRITE(2,420)                                                      CDE04370
  420 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE D2Q(0) FOR N = 1.',        CDE04380
     1 'TAKEN FROM THE FIRST ROW OF THE A6INV MATRIX.')                 CDE04390
      WRITE(2,20) (A6INV(1,J),J=1,6)                                    CDE04400
C                                                                       CDE04410
C*********************************************************************  CDE04420
C                                                                       CDE04430
C         A SUBPROGRAM USED TO COMPUTE THE NONCENTRAL FINITE            CDE04440
C        DIFFERENCE COEFFICIENTS FOR THE DERIVATIVES OF P(R)  AT        CDE04450
C        POINTS NEAR THE BOUNDARIES.  COMPUTE THE AA MATRIX ELEMENTS    CDE04460
C        FOR THE BOUNDARY CONDITIONS NEAR THE AXIS AND WALL.            CDE04470
C        THE APPLICABLE BOUNDARY CONDITIONS FOR N = 1 AND 6 ARE:        CDE04480
C        P(1) = I H(0), A VARIABLE, FOR N = 1 AT THE WALL.              CDE04490
C        P(0) = 0  FOR N = 6 AT THE AXIS.                               CDE04500
C        P(1) = 0  FOR N = 6 AT THE WALL.                               CDE04510
C                                                                       CDE04520
C*********************************************************************  CDE04530
C                                                                       CDE04540
C        COMPUTE THE AA MATRICES AND THEIR INVERSES.                    CDE04550
C                                                                       CDE04560
      DO 500 I=1,5                                                      CDE04570
      FACT = 1.0D0                                                      CDE04580
      XI = DFLOAT(I)                                                    CDE04590
      H(I) = (2.0D0*XI-1.0D0)/2.0D0                                     CDE04600
      DO 500 J=1,5                                                      CDE04610
      XJ = DFLOAT(J)                                                    CDE04620
      FACT = FACT*XJ                                                    CDE04630
      A5(I,J) = (H(I)**J)/FACT                                          CDE04640
  500 CONTINUE                                                          CDE04650
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                           CDE04660
      DO 501 I = 1,4                                                    CDE04670
      DO 501 J = 1,4                                                    CDE04680
      A4(I,J) = A5(I,J)                                                 CDE04690
  501 CONTINUE                                                          CDE04700
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                           CDE04710
C                                                                       CDE04720
C        COMPUTE THE CC COEFFICIENT MATRICES FOR APPROPRIATE POINTS.    CDE04730
```

```
C                                                                       COE04810
      C2X51(1,1) = 1.0D0                                                COE04820
      C2X51(2,1) = 0.0D0                                                COE04830
      FACT = 1.0D0                                                      COE04840
      DO 502 J = 2,5                                                    COE04850
      XJ = DFLOAT(J)                                                    COE04860
      FACT = FACT*(XJ-1.0D0)                                            COE04870
      C2X51(1,J) = (H(1)**(J-1))/FACT                                   COE04880
  502 CONTINUE                                                          COE04890
      DO 503 J = 2,5                                                    COE04900
      JJ = J-1                                                          COE04910
      C2X51(2,J) = C2X51(1,JJ)                                          COE04920
  503 CONTINUE                                                          COE04930
      DO 504 J = 1,4                                                    COE04940
      C1X41(1,J) = C2X51(1,J)                                           COE04950
  504 CONTINUE                                                          COE04960
      C2X52(1,1) = 1.0D0                                                COE04970
      C2X52(2,1) = 0.0D0                                                COE04980
      FACT = 1.0D0                                                      COE04990
      DO 505 J = 2,5                                                    COE05000
      XJ = DFLOAT(J)                                                    COE05010
      FACT = FACT*(XJ-1.0D0)                                            COE05020
      C2X52(1,J) = (H(2)**(J-1))/FACT                                   COE05030
  505 CONTINUE                                                          COE05040
      DO 506 J = 2,5                                                    COE05050
      JJ = J-1                                                          COE05060
      C2X52(2,J) = C2X52(1,JJ)                                          COE05070
  506 CONTINUE                                                          COE05080
      DO 507 J = 1,4                                                    COE05090
      C1X42(1,J) = C2X52(1,J)                                           COE05100
  507 CONTINUE                                                          COE05110
C                                                                       COE05120
CC       PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER        COE05130
CC                                                                      COE05140
CC       CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.          COE05150
C                                                                       COE05160
      CALL VMULFF(C2X51,A5INV,2,5,5,2,5,C25A51,2,IER)                   COE05170
      CALL VMULFF(C2X52,A5INV,2,5,5,2,5,C25A52,2,IER)                   COE05180
      CALL VMULFF(C1X41,A4INV,1,4,4,1,4,C14A41,1,IER)                   COE05190
      CALL VMULFF(C1X42,A4INV,1,4,4,1,4,C14A42,1,IER)                   COE05200
      WRITE(2,508)                                                      COE05210
  508 FORMAT(1X,//)                                                     COE05220
      WRITE(2,509)                                                      COE05230
  509 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQUCOE05240
     1IRED NEAR THE BOUNDARIES FOR N = 1 AND 6.',//,1X,                 COE05250
     2'THE COEFFICIENTS APPEAR IN THE ',                                COE05260
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X, COE05270
     4'THE APPLICABLE BOUNDARY CONDITIONS FOR N = 1 AND 6 ARE:',//,1X,  COE05280
```

```
     5'P(1) = I  H(0),  A VARIABLE,  FOR N = 1 AT THE WALL.',//,1X,        C0E05290
     6'P(0) = 0    FOR N = 5 AT THE AXIS.',//,1X,                          C0E05300
     7'P(1) = 0   FOR N = 5 AT THE WALL.',//)                              C0E05310
      ND = 2                                                               C0E05320
      NP = 1                                                               C0E05330
      NPTS = 5                                                             C0E05340
      WRITE(2,221) ND,NP,NPTS                                              C0E05350
      WRITE(2,20)  (C25A51(2,J),J=1,NPTS)                                  C0E05360
      NP = NP + 1                                                          C0E05370
      WRITE(2,221) ND,NP,NPTS                                              C0E05380
      WRITE(2,20)  (C25A52(2,J),J=1,NPTS)                                  C0E05390
      ND = 1                                                               C0E05400
      NP = 1                                                               C0E05410
      NPTS = NPTS - 1                                                      C0E05420
      WRITE(2,221) ND,NP,NPTS                                              C0E05430
      WRITE(2,20)  (C14A41(1,J),J=1,NPTS)                                  C0E05440
      NP = NP + 1                                                          C0E05450
      WRITE(2,221) ND,NP,NPTS                                              C0E05460
      WRITE(2,20)  (C14A42(1,J),J=1,NPTS)                                  C0E05470
      ND = 9                                                               C0E05480
      NP = 0                                                               C0E05490
      NPTS = 0                                                             C0E05500
      WRITE(2,221) ND,NP,NPTS                                              C0E05510
C                                                                          C0E05520
CC*****************************************************************      C0E05530
CCC                                                                        C0E05540
CCC       A SUBPROGRAM USED TO COMPUTE THE AA MATRIX ELEMENTS FOR          C0E05550
CCC                                                                        C0E05560
CCC       THE BOUNDARY CONDITIONS NEAR THE AXIS FOR N = 1.                 C0E05570
CCC                                                                        C0E05580
CCC       THE APPLICABLE BOUNDARY CONDITION IS:                            C0E05590
CCC                                                                        C0E05600
CCC       DP(0) = 0   FOR N = 1 AT THE AXIS.                               C0E05610
CCC                                                                        C0E05620
CC*****************************************************************      C0E05630
C                                                                          C0E05640
      DO 600 I = 1,5                                                       C0E05650
      A5(I,1) = 1.0D0                                                      C0E05660
  600 CONTINUE                                                             C0E05670
      DO 601 I = 1,4                                                       C0E05680
      A4(I,1) = 1.0D0                                                      C0E05690
  601 CONTINUE                                                             C0E05700
      C2X51(1,1) = 0.0D0                                                   C0E05710
      C2X52(1,1) = 0.0D0                                                   C0E05720
      C1X41(1,1) = 0.0D0                                                   C0E05730
      C1X42(1,1) = 0.0D0                                                   C0E05740
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                              C0E05750
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                              C0E05760
```

```
C                                                                       COE05770
C       PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER         COE05780
C                                                                       COE05790
C       CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.           COE05800
C                                                                       COE05810
      CALL VMULFF(C2X51,A5INV,2,5,5,2,5,C25A51,2,IER)                   COE05820
      CALL VMULFF(C2X52,A5INV,2,5,5,2,5,C25A52,2,IER)                   COE05830
      CALL VMULFF(C1X41,A4INV,1,4,4,1,4,C14A41,1,IER)                   COE05840
      CALL VMULFF(C1X42,A4INV,1,4,4,1,4,C14A42,1,IER)                   COE05850
      WRITE(2,605)                                                      COE05860
  605 FORMAT(1X,//)                                                     COE05870
      WRITE(2,606)                                                      COE05880
  606 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQU COE05890
     1IRED NEAR THE AXIS FOR N = 1 .',//,1X,                            COE05900
     2'  THE COEFFICIENTS APPEAR IN THE ',                             COE05910
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X, COE05920
     4'THE APPLICABLE BOUNDARY CONDITION IS:',//,1X,                    COE05930
     5'DP(0) = 0  FOR N = 1 AT THE AXIS.',//)                           COE05940
      ND = 2                                                            COE05950
      NP = 1                                                            COE05960
      NPTS = 5                                                          COE05970
      WRITE(2,221) ND,NP,NPTS                                           COE05980
      WRITE(2,20)  (C25A51(2,J),J=1,NPTS)                               COE05990
      NP = NP + 1                                                       COE06000
      WRITE(2,221) ND,NP,NPTS                                           COE06010
      WRITE(2,20)  (C25A52(2,J),J=1,NPTS)                               COE06020
      ND = 1                                                            COE06030
      NP = 1                                                            COE06040
      NPTS = NPTS - 1                                                   COE06050
      WRITE(2,221) ND,NP,NPTS                                           COE06060
      WRITE(2,20)  (C14A41(1,J),J=1,NPTS)                               COE06070
      NP = NP + 1                                                       COE06080
      WRITE(2,221) ND,NP,NPTS                                           COE06090
      WRITE(2,20)  (C14A42(1,J),J=1,NPTS)                               COE06100
      ND = 9                                                            COE06110
      NP = 0                                                            COE06120
      NPTS = 0                                                          COE06130
      WRITE(2,221) ND,NP,NPTS                                           COE06140
      WRITE(2,615)                                                      COE06150
  615 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE P(0) FOR N = 1.',          COE06160
     1 'TAKEN FROM THE FIRST ROW OF THE A5INV MATRIX.')                 COE06170
      WRITE(2,20) (A5INV(1,J),J=1,5)                                    COE06180
      WRITE(2,616)                                                      COE06190
  616 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE D2P(0) FOR N = 1.',         COE06200
     1 'TAKEN FROM THE SECOND ROW OF THE A5INV MATRIX.')                COE06210
      WRITE(2,20) (A5INV(2,J),J=1,5)                                    COE06220
      STOP                                                              COE06230
      END                                                               COE06240
```

```
*
*  FILENAME:   RUNDATA EXEC
*
*  THIS EXEC ALLOWS THE USER TO RUN A FORTRAN H EXTENDED PROGRAM ON
*
*  VM/CMS THAT REQUIRES INPUT DATA AND/OR PRODUCES OUTPUT DATA.
*
*  THE INPUT DATA IS STORED ON THE USER'S DISK UNDER THE SAME FILENAME
*
*  ASSIGNED TO THE MAIN PROGRAM WITH FILETYPE = <DATAIN>.  THE MAIN
*
*  PROGRAM MUST USE, READ(3,NNN) ,I.E. DEVICE CODE 3.  THE OUTPUT DATA
*
*  FROM THE FORTRAN PROGRAM WILL BE SENT TO A FILE ON THE USER'S DISK
*
*  HAVING THE SAME FILENAME WITH FILETYPE = <DATAOUT2>.  USE
*
*  WRITE(2,NNN)  , I.E. DEVICE CODE 2.  THE USER ALSO HAS THE CAPABILITY
*
*  TO LOAD AND EXECUTE PRE-COMPILED, (FILETYPE = TEXT), SUBROUTINES
*
*  THAT RESIDE ON THE USER'S DISK WHICH ARE CALLED BY THE MAIN PROGRAM.
*
*  TYPE THE COMMAND:  RUNDATA  <FN>
*
*  THE OUTPUT IS SENT TO DISK, A FILE IS CREATED AS <FN> <DATAOUT2>.
*
*  IF DEVICE CODE 6 IS also USED IN THE MAIN PROGRAM, I.E. WRITE(6,NNN),
*
*  THE OUTPUT WILL AUTOMATICALLY GO TO THE TERMINAL AS WELL.
*
FORTHX &1
FILEDEF 03 DISK &1 DATAIN (LRECL 80
FILEDEF 02 DISK &1 DATAOUT2 (BLOCK 130
FILEDEF 06 TERMINAL
LOAD &1 (START
ERASE &1 TEXT *
ERASE &1 LISTING *
ERASE LOAD MAP
&EXIT
```

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A numerical analysis of pipe flow stability.